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NON-HOMOGENEOUS SQUARE FUNCTIONS ON GENERAL SETS: SUPPRESSION AND BIG PIECES METHODS

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ABSTRACT. We aim to showcase the wide applicability and power of the big pieces and suppression methods in the theory of local Tb theorems. The setting is new: we consider conical square functions with cones $\{x \in \mathbb{R}^n \setminus E : |x-y| < 2\operatorname{dist}(x,E)\}$, $y \in E$, defined on general closed subsets $E \subset \mathbb{R}^n$ supporting a non-homogeneous measure μ. We obtain boundedness criteria in this generality in terms of weak type testing of measures on regular balls $B \subset E$, which are doubling and of small boundary. Due to the general set E we use metric space methods. Therefore, we also demonstrate the recent techniques from the metric space point of view, and show that they yield the most general known local Tb theorems even with assumptions formulated using balls rather than the abstract dyadic metric cubes.

1. INTRODUCTION

Square functions are important objects in harmonic analysis and in the theory of PDEs. For example, think of the characterisation of classical Hardy spaces in terms of vertical and conical square function estimates for Littlewood-Paley operators [19], or the characterisation of the uniformly rectifiable sets in terms of square function estimates of the double gradient of the single layer potential associated with the Laplace operator [6]. We also consider them important from the framework point of view: they aid in developing novel techniques that work for singular integrals. In this paper we are interested in a new setup, but also in analysing the very latest *methods of proof* and *characterisations* of the boundedness of square functions (or singular integrals) in the context of non-homogeneous analysis.

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For us characterisations of boundedness means various types of sophisticated Tb theorems – mainly of *local* and also of the so called *big pieces* type. The latter big pieces type is not equally well-known, but has featured prominently in many recent important articles, for example in connection with the breakthroughs related to the rectifiability of the harmonic measure [5]. Such a big pieces Tb was first proved by Nazarov, Treil and Volberg [17] (for Cauchy integral type operators) in connection with Vitushkin's conjecture. Their point is roughly the following: the assumptions are much weaker than in the usual Tb theorems, but so are the conclusions in that one only gets the boundedness of the given operator on some big piece. These theorems have lately become more and more important, and there is also a highly useful connection between them and the local Tb theorems, as observed by us in [15]. For a relatively detailed account of the entangled history of the local Tb theorems we refer to [15].

Lowering the integrability of the appearing test functions is the key problem in local Tb theorems, see e.g. [1], [3], [4], [8], [11], [14], [15]. Related to this and other issues, we developed in [15] a new method to prove local Tb theorems via the big pieces and good lambda methods. Let us explain the main steps of our method later. What it allows, however, is weak type $(1,1)$ testing conditions and improving known results – both doubling and non-doubling – in other ways. The method is important regarding Calderón–Zygmund operators too, see [14] for the best known integrability exponents related to Hofmann's problem. A major technical convenience of the method is that the difficult core Tb argument can be carried out in the big pieces Tb part, and is hence of L^2 nature even if the original test functions are not.

In this paper we give the full technical execution of our method in the novel setting of conical square functions defined on general closed subsets $E \subset \mathbb{R}^n$ supporting a non-homogeneous measure μ. This is the first time non-homogeneous analysis is being carried out in this particular context. It is an interesting setting with new difficulties of its own, but also provides us an opportunity to show a full breakdown of the required components in a very general, essentially metric in nature, setting. The setup includes two measures: one on E and one on the complement $\mathbb{R}^n \setminus E$. Related Ahlfors–David regular theory has been developed in e.g. [9] by Hofmann, Mitrea, Mitrea and Morris and in [7] by one of us and A. Herran. Our methods are completely different, not only because of the non-homogeneous setting, but also because we prove more advanced local Tb theorems using the latest methods of [15]. In this paper the measure μ living on E can be non-doubling, but the measure used to integrate over the complement $\mathbb{R}^n \setminus E$ is just an appropriately weighted Lebesgue measure. For the exact setting see Section 1.1.

Let us still highlight an additional benefit of our method that surfaces from this paper. It is the following technical aspect related especially to the metric space theory of local Tb theorems. With the most general possible local Tb theorems (particularly with Calderón–Zygmund operators, general integrability exponents or non-homogeneous measures) the passage between having the test functions b_Q on some (completely abstract) metric dyadic cubes Q or having the test functions b_B on metric balls B does not appear to be straightforward, see e.g. the paper by Auscher–Routin [3]. The reasons are technical: the operator testing condition does not always seem to be trivial to transfer (from balls to cubes) with general L^p integrability exponents. This is because the Hardy inequality might not be available due to some irregular underlying metric measure space, or even if it is, it is not useful with certain exponent (which is called the super-dual case, see e.g. [3]). Even the accretivity condition is a problem with non-homogeneous measures. But the old proofs seem to require the existence of b_Q on dyadic cubes (this is required to build some dyadic martingales). For these reasons some of these theorems have been previously formulated involving these rather abstract cubes even in the statements. A point we like to make is that these new methods are flexible enough in that we can easily state and prove our extremely general local Tb having the existence of the test functions b_B only on very regular balls B.

1.1. **The setting.** Let E be a closed subset of $\mathbb{R}^n$ and $S\colon \mathbb{R}^n \setminus E \times E \to \mathbb{C}$ be a kernel, which, for some fixed $\alpha, \beta \in (0,1]$, $m > 0$, and $K_1, K_2 > 0$, satisfies the size estimate

$$|S(x,y)| \le K_1 \frac{1}{|x-y|^{m+\alpha}}, \tag{1.1}$$

for all $(x,y) \in \mathbb{R}^n \setminus E \times E$, and the y-Hölder estimate

$$|S(x,y) - S(x,y')| \le K_2 \frac{|y-y'|^{\beta}}{|x-y|^{m+\alpha+\beta}}, \tag{1.2}$$

where $x \in \mathbb{R}^n \setminus E, y, y' \in E$ and $|y-y'| \le \frac{|x-y|}{2}$. If μ is a non-negative finite measure with support in E, or if μ is a measure of order m in E (see definition below), we define the integral operator T_μ on $\bigcup_{p \in [1,\infty]} L^p(\mu)$ by

$$T_\mu f(x) := \int_E S(x,z) f(z) \, \mathrm{d}\mu(z), \quad x \in \mathbb{R}^n \setminus E.$$

Note that the integral is absolutely convergent. Let σ be a measure in $\mathbb{R}^n \setminus E$ given by

$$\sigma(A) := \int_A \frac{\mathrm{d}m_n(x)}{d(x,E)^n},$$

where $A \subset \mathbb{R}^n \setminus E$ is any Lebesgue measurable set. Here $d(x,E)$ stands for the distance of the point x to E and m_n is the Lebesgue measure in $\mathbb{R}^n$. For a point $y \in E$, we denote the *cone* $\Gamma(y)$ at y by

$$\Gamma(y) := \big\{ x \in \mathbb{R}^n \setminus E : |x-y| < 2d(x,E) \big\}.$$

We can now define the *conical square function* $\mathcal{C}_\mu$ on $\bigcup_{p\in[1,\infty]} L^p(\mu)$ by setting

$$\mathcal{C}_\mu f(y) := \Big(\int_{\Gamma(y)} |T_\mu f(x)|^2 d(x,E)^{2\alpha} \,\mathrm{d}\sigma(x) \Big)^{\frac{1}{2}}, \quad y \in E.$$

Note that we will also use square functions with other measures than μ. If $\mathcal{M}(E)$ is the set of complex measures with support in E, then for $\nu \in \mathcal{M}(E)$, $x \in \mathbb{R}^n \setminus E$ and $y \in E$ we define

$$T\nu(x) := \int_E S(x,z)\,\mathrm{d}\nu(z)$$

and

$$\mathcal{C}\nu(y) := \Big(\int_{\Gamma(y)} |T\nu(x)|^2 d(x,E)^{2\alpha} \,\mathrm{d}\sigma(x) \Big)^{\frac{1}{2}}.$$

We will need to use various truncated versions of the square function. For any $t > 0$ and $y \in E$ define the truncated cones

$$\Gamma_t(y) := \{x \in \Gamma(y) \colon d(x,E) > t\} \quad \text{and} \quad \Gamma^t(y) := \{x \in \Gamma(y) \colon d(x,E) \le t\}.$$

If $0 < s < t$ we set $\Gamma_s^t(y) := \Gamma_s(y) \cap \Gamma^t(y)$. The corresponding truncated square functions, defined with integration over the truncated cones only, are denoted for instance by $\mathcal{C}^t$ or $\mathcal{C}^t_{\mu,s}$, depending on the situation. As an example we have

$$\mathcal{C}^t_\mu f(y) := \Big(\int_{\Gamma^t(y)} |T_\mu f(x)|^2 d(x,E)^{2\alpha} \,\mathrm{d}\sigma(x) \Big)^{\frac{1}{2}}, \quad y \in E.$$

Before going any further, let us introduce some notation which is necessary for the statement of our main theorem.

1.2. **Notation and key definitions.** An open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x,r)$, while an open ball in E with center $y \in E$ and radius $r > 0$ is defined as $B_E(y,r) := B(y,r) \cap E$. Often, when there is no danger of confusion, we may drop the subscript E. We write $\bar{B}(x,r)$ and $\bar{B}_E(y,r)$ for the corresponding closed balls and $r(B)$ for the radius of the ball B. If we talk about a ball without specifying whether it is open or closed, then it should be understood that it can be either one.

For two constants $A, B \ge 0$ the notation $A \lesssim B$ means that there exists an absolute constant $C > 0$ such that $A \le CB$. We write $A \lesssim_{\alpha,\beta,\dots} B$ to indicate the dependence of the constant C on α, β, etc. Two sided estimates $A \lesssim B \lesssim A$ are abbreviated as $A \sim B$.

If $\nu \in \mathcal{M}(E)$, then $|\nu|$ denotes its *total variation*. We say that a measure μ is *of order m in E* if μ is non-negative, $\operatorname{spt}\mu \subset E$ and

$$\mu(B(y,r)) \lesssim r^m \tag{1.3}$$

holds for all $y \in E$ and $r > 0$. If μ is a non-negative Borel measure or a complex Borel measure in $\mathbb{R}^n$ and $F \subset \mathbb{R}^n$ is a Borel set, then $\mu\lfloor F$ is a measure defined by $\mu\lfloor F(G) = \mu(F \cap G)$, where G is any Borel set.

Let $a, b \geq 1, \kappa > 0$ and suppose μ is a non-negative Borel measure in $\mathbb{R}^n$. A ball $B(x_0, r)$ is said to be (a, b)-*doubling* (with respect to μ) if

$$\mu(B(x_0, ar)) \leq b\mu(B(x_0, r)). \tag{1.4}$$

The ball $B(x_0, r)$ is said to have κ-small boundary if for all $s \in [0, 1]$ there holds that

$$\mu\big(\{x \in \mathbb{R}^n \colon r(1-s) < |x - x_0| < r(1+s)\}\big) \leq \kappa s\, \mu(B(x_0, 3r)). \tag{1.5}$$

We say that a ball $B_E(y, r)$ is (a, b)-doubling or that it has κ-*small boundary* if $B(y, r)$ has the corresponding property. These concepts are defined similarly with closed balls just by replacing the open balls in (1.4) and (1.5) with closed balls (the left hand side of (1.5) stays the same).

1.3. **Statement of the main theorem and further discussion.** We will next formulate our main local Tb theorem for $\mathcal{C}_\mu$. First, for experts and non-experts alike let us try to identify the main points:

- A testing measure ν_B is given on each *regular* (i.e. doubling and of small boundary) surface *ball* $B \subset E$.
- The testing measure (or somewhat less generally the testing function) ν_B is required to be supported on B and accretive (normalised to the condition $\nu_B(B) = \mu(B)$). What is important is that it need not satisfy strong estimates, only the L^1 type condition $|\nu_B|(B) \leq C_1\mu(B)$ and some rather weak *quantified* absolute continuity assumption (assumption 4) below). For example, the latter technical condition is automatically satisfied by Hölder's inequality should the measure be a function $\nu_B = b_B\, d\mu$ satisfying a slightly stronger $L^{1+\epsilon}$ type condition. That is why it does not appear in more classical formulations.
- The testing condition on the operator side is completely decoupled from the regularity assumption on ν_B. We only require the weak type estimate $\sup_{\lambda>0} \lambda^s \mu\big(\{y \in B : \mathcal{C}^{r(B)}\nu_B(y) > \lambda\}\big) \leq C_2|\nu_B|(B)$ for some $s > 0$ (e.g. $s = 1$). In fact, we may only require this estimate outside some small enough exceptional set $U_B \subset B$.
- Using weak type testing is also in line with keeping the condition necessary for the boundedness: L^2 boundedness implies such a condition with $s = 1$ using the boundedness of $\mathcal{C}$ from the set of finite measures to $L^{1,\infty}(\mu)$.

1.6. **Theorem.** *Suppose μ is a measure of order m in E. Let $b, \kappa > 0$ be big enough constants depending only on the dimension n, and let $\varepsilon_0 \in (0, 1)$ and $C_1 > 0$ be given constants. Assume that for every $(10, b)$-doubling closed ball B in E with a κ-small boundary (with respect to μ) there exists a complex measure $\nu_B \in \mathcal{M}(E)$ with the following properties:*

(1) $\operatorname{spt} \nu_B \subset B$;
(2) $\nu_B(B) = \mu(B)$;

(3) $|\nu_B|(B) \le C_1\mu(B)$*;*
(4) If $A \subset B$ *is a Borel set with* $\mu(A) \le \varepsilon_0\mu(B)$*, then* $|\nu_B|(A) \le \frac{1}{16C_1}|\nu_B|(B)$*.*

Furthermore, assume that we are given constants $s, C_2 > 0$*, and in every ball* B *as above a set* $U_B \subset B$*, so that*
a) $|\nu_B|(U_B) \le \frac{1}{16C_1}|\nu_B|(B)$*;*
b) $\sup_{\lambda>0} \lambda^s \mu\big(\{y \in B \setminus U_B : \mathcal{C}^{r(B)}\nu_B(y) > \lambda\}\big) \le C_2|\nu_B|(B)$*.*

Under these assumptions the square function $\mathcal{C}_\mu$ *is bounded in* $L^p(\mu)$ *for every* $p \in (1,\infty)$ *with norm depending on* p *and the preceding constants.*

1.7. *Remark.* When $\operatorname{diam}(E) < \infty$ one can restrict the testing surface balls to have radius $r \in (0, \operatorname{diam}(E)]$. But then one needs to interpret E to be one of the balls so that ν_E exists.

Such a theorem was obtained in [15] for vertical square functions in the upper half space $\mathbb{R}^{n+1}_+$ and in [14] for maximal truncations of Calderón-Zygmund operators. Although the method is the same, to prove Theorem 1.6 in this generality, one has to overcome non-trivial issues stemming from the geometrically complicated environment on which our objects are defined.

It is not completely evident which measure σ one should or could use on the complement $\mathbb{R}^n \setminus E$ to define the integration over the cones $\Gamma(y) \subset \mathbb{R}^n \setminus E$, $y \in E$. In [13] we used $\sigma = \mu \times dt/t^{m+1}$ with conical square functions in the upper half-space $\mathbb{R}^{k+1}_+$ and with a measure μ of order m supported in $\mathbb{R}^k$ (this is essentially the setting $n = k+1$ and $E = \mathbb{R}^k \times \{0\}$). A key requirement is that σ ought to be a measure on $\mathbb{R}^n \setminus E$ that assigns a bounded measure to Whitney regions associated with E (like the measure $\sigma = \mu \times dt/t^{m+1}$ in the upper-half space does: $\sigma(B \times (r(B)/2, r(B)) \lesssim \mu(B)/r(B)^m \lesssim 1)$. For some parts of the theory there is another important aspect in play: we also need to be able to exploit the fact that nearby cones have some geometric cancellation and then see this appropriately on the σ measure side (this also works with $\sigma = \mu \times dt/t^{m+1}$ in the upper half-space because of the appropriate Lebesgue part). For this latter reason we use just the Lebesgue measure weighted with $d(x,E)^{-n}$ here i.e. for us $\sigma = d(x,E)^{-n}\,dm_n$ like defined above. As is apparent from the upper half-space situation, in some scenarios multiple other natural choices also work. Further investigation on this issue could be appropriate – but for us the main thing was to remove regularity assumptions on μ. Compare also to [9] where two different ADR measures are used.

1.4. **Outline of the method and proof.** The key steps of our method are as follows:

(1) The non-homogeneous good lambda method by Tolsa [20]. This can simply be thought of as a highly flexible way to glue some local results in to the desired global result. It says that it is enough to find in every $(10,b)$-doubling closed ball B in E with a κ-small boundary a subset $G_B \subset B$

with $\mu(G_B) \gtrsim \mu(B)$, where

$$\mathcal{C}\colon \mathcal{M}(E) \to L^{1,\infty}(\mu\lfloor G_B)$$

is bounded with a constant C that is independent of B, that is, the inequality

$$\mu\big(\{y \in G_B \colon \mathcal{C}\nu(y) > \lambda\}\big) \le \frac{C}{\lambda}|\nu|(E)$$

holds for all $\lambda > 0$ and $\nu \in \mathcal{M}(E)$. We give a proof in our context in Section 5.

(2) Prove a big pieces global Tb: Theorem 6.1. This is interesting on its own right but is also the key step in proving the required local result needed to apply the non-homogeneous good lambda method. The theorem says that given a closed ball B, a finite measure μ with support in B and a function $b \in L^\infty(\mu)$, then under certain very weak assumptions one finds a subset $G \subset B$ with $\mu(G) \gtrsim \mu(B)$ so that the square function $\mathcal{C}_\mu$ is bounded in $L^2(\mu\lfloor G_B)$. The proof relies on the idea of *suppression* – an amazing technique of Nazarov–Treil–Volberg designed to cook up an operator that behaves significantly better than the original one but also agrees with the original one on a large enough set. In this paper we show how to suppress these conical square functions.

(3) The third step is to prove that the assumptions of the non-homogeneous good lambda method hold in our situation. This is Proposition 3.1. So we take an arbitrary closed $(10, b)$-doubling ball B in E with a κ-small boundary. Then, by assumption, we have the test measure ν_B. By writing the polar decomposition $\nu_B = b|\nu_B|$, where $|b(y)| = 1$ for all $y \in B$, we have

$$\mathcal{C}\nu_B = \mathcal{C}_{|\nu_B|}b.$$

Next, we want to use the big pieces global Tb theorem with the measure $|\nu_B|$ and the bounded function b. Using stopping times we can do this, and we get a set G_B where $\mathcal{C}_{|\nu_B|}$ is bounded in $L^2(|\nu_B|\lfloor G_B)$. This will be done in such a way that $|\nu_B|\lfloor G_B = \varphi\mu\lfloor G_B$ for some function $\varphi \sim 1$, whence we can come back to the measure μ and derive the desired result that $\mathcal{C}_\mu$ is bounded in $L^2(\mu\lfloor G_B)$. A weak $(1,1)$ argument is required to conclude that the assumptions of the good lambda theorem hold.

We also need various preliminaries (Section 2). Some geometric considerations related to cones are presented in Section 4. Certain technical details are also given in the Appendixes. We use metric dyadic cubes and their randomisation purely as a technical tool (the cubes do not appear in the statement of the main theorem). This is needed for the Tb argument in the big pieces Tb theorem. The randomisation originates from the paper by Hytönen–Martikainen [10] but we use the latest version from Auscher–Hytönen [2] with some further modifications.

2. Preliminaries

We begin by collecting here some notation and basic estimates that will be used in the later sections. We shall use maximal functions on the set E. Let μ be a non-negative locally finite Borel measure in E. The *centred* Hardy-Littlewood maximal operator is defined for any non-negative locally finite Borel measure ν with support in E by

$$M_\mu \nu(y) := \sup_{r>0} \frac{\nu(B(y,r))}{\mu(B(y,r))}, \quad y \in E,$$

and the *radial* one by

$$M^m \nu(y) := \sup_{r>0} \frac{\nu(B(y,r))}{r^m}, \quad y \in E.$$

If $\nu \in \mathcal{M}(E)$, we set $M_\mu \nu := M_\mu |\nu|$ and $M^m \nu := M^m |\nu|$. If $f \in L^1_{loc}(\mu)$ we write $M_\mu f := M_\mu(|f|\mu)$.

A basic property of M_μ is that it is bounded from $\mathcal{M}(E)$ into $L^{1,\infty}(\mu)$ and from $L^p(\mu)$ into $L^p(\mu)$, $p \in (1,\infty)$. If μ is of order m there holds that $M^m\nu(y) \lesssim M_\mu \nu(y)$ for every y and $\nu \in \mathcal{M}(E)$, and therefore the boundedness properties of M_μ transfer to M^m.

2.1. **Lemma.** *Let $y \in E$, $r > 0$ and suppose μ is a non-negative Borel measure with* $\operatorname{spt} \mu \subset E$. *If $f \in L^1_{loc}(\mu)$, then*

$$\int_E \frac{|f(z)| \, d\mu(z)}{(r + |z-y|)^{m+\alpha}} \lesssim r^{-\alpha} M^m(|f|\mu)(y),$$

where the implicit constant is independent of r and y.

The proof of Lemma 2.1 is a standard calculation dividing the integration area as

$$E = \big(B_E(y,r)\big) \cup \bigcup_{k=0}^{\infty} \big(B_E(y, 2^{k+1}r) \setminus B_E(y, 2^k r)\big).$$

The next lemma is a simple geometric observation:

2.2. **Lemma.** *Let $y, z \in E$ and $x \in \Gamma(y)$. Then*

$$|x-z| \sim |x-y| + |y-z|.$$

Proof. If $|y-z| \geq 2|x-y|$, then

$$|x-z| \geq |y-z| - |x-y| \sim |y-z| + |x-y|.$$

On the other hand if $|y-z| < 2|x-y|$, then

$$|x-z| \geq d(x,E) > \frac{1}{2}|x-y| \sim |x-y| + |y-z|.$$

□

2.3. **Lemma.** *For every $y \in E$ and $t > s > 0$ there holds that*

$$\sigma\big(\Gamma^t_s(y)\big) \lesssim \left(\frac{t}{s}\right)^n.$$

Proof. Fix some $y \in E$ and $t > s > 0$. If $x \in \Gamma^t_s(y)$, then we have

$$|x-y| < 2d(x,E) \le 2t.$$

Also, x satisfies $d(x,E) \ge s$. From these we directly get that

$$\sigma\big(\Gamma^t_s(y)\big) = \int_{\Gamma^t_s(y)} \frac{\mathrm{d}m_n(x)}{d(x,E)^n} \le \frac{m_n(B(y,2t))}{s^n} \sim \left(\frac{t}{s}\right)^n.$$

□

From Lemma 2.3 we get the following two lemmas:

2.4. **Lemma.** *Let $s,t > 0$. Then for every $y \in E$ and $r > 0$ there holds that*

$$\int_{\Gamma(y)} \frac{d(x,E)^s}{(d(x,E)+r)^{s+t}} \,\mathrm{d}\sigma(x) \lesssim_{s,t} r^{-t}.$$

Proof. Fix some $y \in E$ and $r > 0$. Then

$$\int_{\Gamma^r(y)} \frac{d(x,E)^s}{(d(x,E)+r)^{s+t}} \,\mathrm{d}\sigma(x) \le \sum_{k=0}^{\infty} \sigma\big(\Gamma^{2^{-k}r}_{2^{-k-1}r}(y)\big) \frac{(2^{-k}r)^s}{r^{s+t}} \lesssim_s \frac{1}{r^t}$$

and

$$\int_{\Gamma_r(y)} \frac{d(x,E)^s}{(d(x,E)+r)^{s+t}} \,\mathrm{d}\sigma(x) \le \sum_{k=0}^{\infty} \sigma\big(\Gamma^{2^{k+1}r}_{2^k r}(y)\big) \frac{1}{(2^k r)^t} \lesssim_t \frac{1}{r^t}.$$

□

2.5. **Lemma.** *Suppose μ is a measure of order m in E. For every $t > s > 0$ and $p \in (1,\infty)$ the truncated square function $\mathcal{C}^t_{\mu,s}$ is bounded in $L^p(\mu)$.*

Proof. Fix some numbers $0 < s < t$ and $p \in (1,\infty)$, and suppose $f \in L^p(\mu)$. If $y \in E$ and $x \in \Gamma(y)$, then Lemmas 2.1 and 2.2 give

$$\begin{aligned} |T_\mu f(x)| &\lesssim \int_E \frac{|f(z)|}{|x-z|^{m+\alpha}} \,\mathrm{d}\mu(z) \sim \int_E \frac{|f(z)|}{(|x-y|+|y-z|)^{m+\alpha}} \,\mathrm{d}\mu(z) \\ &\lesssim |x-y|^{-\alpha} M^m(|f|\mu)(y) \\ &\lesssim |x-y|^{-\alpha} M_\mu f(y). \end{aligned} \tag{2.6}$$

Combining this with Lemma 2.3 yields

$$\begin{aligned} \mathcal{C}^t_{\mu,s} f(y) = \Big(\int_{\Gamma^t_s(y)} |T_\mu f(x)|^2 d(x,E)^{2\alpha} \,\mathrm{d}\sigma(x) \Big)^{\frac{1}{2}} &\lesssim \sigma(\Gamma^t_s(y))^{\frac{1}{2}} M_\mu f(y) \\ &\lesssim_{s,t} M_\mu f(y). \end{aligned}$$

Thus $\mathcal{C}^t_{\mu,s} f$ is pointwise dominated by M_μ, and the claim follows from boundedness of M_μ in $L^p(\mu)$. □

Random dyadic cubes. In the big piece global Tb theorem and in the Whitney decomposition related to the good lambda method we shall use systems of dyadic cubes in E. Even though we are in $\mathbb{R}^n$, the existence of these is a metric space argument, because there is no direct way of building dyadic systems in *arbitrary* closed subsets of $\mathbb{R}^n$. The specific construction we use is from [2], and we elaborate here on how we use it.

Let $x \in E$ be a fixed point. The construction of dyadic cubes begins by choosing a set of *reference dyadic points*. Write $\delta = \frac{1}{1000}$. Set $\mathscr{X}_0$ to be any maximal 1-separated subset of E so that $x \in \mathscr{X}_0$. Let $K \in \mathbb{Z}, K \geq 0$, and assume that the sets $\mathscr{X}_k$ and $\mathscr{X}_{-k}$ have been chosen for $k \in \{0, \ldots, K\}$. Define $\mathscr{X}_{K+1}$ to be any maximal δ^{K+1}-separated subset of E such that $\mathscr{X}_K \subset \mathscr{X}_{K+1}$, and $\mathscr{X}_{-K-1}$ to be any maximal δ^{-K-1}-separated subset of $\mathscr{X}_{-K}$ so that $x \in \mathscr{X}_{-K-1}$. Continue this way to get the collections $\mathscr{X}_k$ for $k \in \mathbb{Z}$. Then $\mathscr{X} := \bigcup_k \mathscr{X}_k$ is the set of reference dyadic points. A point in $\mathscr{X}$ is denoted by x^k_α, where k indicates that $x^k_\alpha \in \mathscr{X}_k$, and α indexes the different points in $\mathscr{X}_k$. This is precisely as in [2], except that here we require that $x \in \mathscr{X}_k$ for every k. The role of this fixed point will be explained below.

The rest of the construction we take directly as in [2]. We have a probablity space $\Omega = (\{0, \ldots, L\} \times \{1, \ldots, M\})^{\mathbb{Z}}$, where the numbers L and M are related to the properties of E as a geometrically doubling space. That E is geometrically doubling means that there exists a constant N such that every ball B in E with radius r contains at most N points whose distances from each other are at least $r/2$. The set Ω is equipped with the natural σ-algebra and the probability measure $\mathbb{P}$ so that the coordinate mappings

$$\omega \mapsto \omega(k) \in \{0, \ldots, L\} \times \{1, \ldots, M\},$$

where $k \in \mathbb{Z}$, are independent and uniformly distributed over the finite set $\{0, \ldots, L\} \times \{1, \ldots, M\}$.

With every $\omega \in \Omega$ there is associated a set $\{z^k_\alpha(\omega)\}_{k,\alpha}$ of slightly shifted reference dyadic points. To every $z^k_\alpha(\omega)$ corresponds a dyadic cube $Q^k_\alpha(\omega) \subset E$, and $\mathcal{D}_k(\omega)$ is the collection all cubes $Q^k_\alpha(\omega)$ with the fixed generation k. The dyadic lattice $\mathcal{D}(\omega)$ is $\bigcup_{k \in \mathbb{Z}} \mathcal{D}_k(\omega)$. To be precise, in [2] certain open and closed dyadic cubes are constructed, and from these we form our cubes using finite unions and intersections as is done in [10], Theorem 4.4.

We list a few relevant properties of the dyadic systems and introduce some notation that will be used later. Let $\omega \in \Omega$.

- Every dyadic cube $Q^k_\alpha(\omega)$ is a measurable subset of E such that

$$B_E(z^k_\alpha(\omega), \frac{\delta^k}{6}) \subset Q^k_\alpha(\omega) \subset B_E(z^k_\alpha(\omega), 6\delta^k). \tag{2.7}$$

We call the point $z^k_\alpha(\omega)$ the center of $Q^k_\alpha(\omega)$, and we set $\ell(Q^k_\alpha(\omega)) := \delta^k$ to be the "sidelength" of $Q^k_\alpha(\omega)$. Also, the original reference points satisfy

$$B_E(x^k_\alpha, \frac{\delta^k}{8}) \subset Q^k_\alpha(\omega) \subset B_E(x^k_\alpha, 8\delta^k). \tag{2.8}$$

- The cubes in a given generation are pairwise disjoint and cover the whole set E, that is, $Q^k_\alpha(\omega) \cap Q^k_\beta(\omega) = \emptyset$ if $\alpha \neq \beta$ and $E = \bigcup_\alpha Q^k_\alpha(\omega)$ for every $k \in \mathbb{Z}$.
- The dyadic cubes are nested in the sense that for any two cubes $Q^k_\alpha(\omega)$ and $Q^l_\beta(\omega)$ one of the following holds: $Q^k_\alpha(\omega) \cap Q^l_\beta(\omega) = \emptyset, Q^k_\alpha(\omega) \subset Q^l_\beta(\omega)$ or $Q^l_\beta(\omega) \subset Q^k_\alpha(\omega)$.
- If $Q^k_\alpha(\omega) \in \mathcal{D}(\omega)$ we denote by $\mathrm{ch}(Q^k_\alpha(\omega))$ the collection of cubes $Q^{k+1}_\beta \in \mathcal{D}_{k+1}(\omega)$ such that $Q^{k+1}_\beta(\omega) \subset Q^k_\alpha(\omega)$. These are called *children* of $Q^k_\alpha(\omega)$.
- $\widehat{Q^k_\alpha}(\omega)$ denotes the unique cube in $\mathcal{D}_{k-1}(\omega)$ that contains $Q^k_\alpha(\omega)$.
- In the rest of the paper we usually write Q in place of $Q^k_\alpha(\omega)$. Nevertheless, one should keep in mind that this is only a short hand, and there is always the specified generation k such that $Q \in \mathcal{D}_k(\omega)$. This is important because it may happen that $Q^k_\alpha(\omega) = Q^l_\beta(\omega)$ as *sets in* E even if $k \neq l$. In the summations over dyadic cubes below, we are always summing over pairs (k, α). If $Q = Q^k_\alpha(\omega)$, then its center $z^k_\alpha(\omega)$ will be denoted by c_Q.
- Let B be a ball in E with center $y \in E$ and radius r. Construct the random dyadic systems $\mathcal{D}(\omega)$ in E with the initial requirement that $y \in \bigcap_{k\in\mathbb{Z}} \mathscr{X}_k$. For any $k \in \mathbb{Z}$ there exists $\alpha(k)$ so that $y = x^k_{\alpha(k)}$. Specify $k_0 \in \mathbb{Z}$ by the condition $r < \frac{\delta^{k_0}}{8} \leq \delta^{-1} r$. Then define

  $$Q_B(\omega) := Q^{k_0}_{\alpha(k_0)}(\omega)$$

  and

  $$\mathcal{D}_B(\omega) := \big\{ Q^l_\beta(\omega) \in \mathcal{D}(\omega) : l \geq k_0, Q^l_\beta(\omega) \subset Q_B(\omega) \big\}. \tag{2.9}$$

  With these definitions we have $B \subset Q_B(\omega)$ by (2.8).

We shall also use a variant of the notion of *good dyadic cubes* introduced first by Nazarov, Treil, and Volberg [18], and then used in the metric space setting for instance in [10]. Let again B be some ball in E. Using the center of B as the fixed reference dyadic point construct the dyadic systems $\mathcal{D}_B(\omega) \subset \mathcal{D}(\omega), \omega \in \Omega$, as described above. Fix some $\omega_0 \in \Omega$. Let $\gamma = \frac{\alpha}{2(m+\alpha)}$ and $r \in \mathbb{Z}, r > 0$. A cube $R \in \mathcal{D}(\omega_0) =: \mathcal{D}_0$ is said to be $\mathcal{D}_B(\omega)$-*good* (with parameters (γ, r)) if for all $Q \in \mathcal{D}_B(\omega)$ with $\ell(Q) \geq \delta^{-r}\ell(R)$ there holds that

$$\max\big(d(R, Q), d(R, E \setminus Q)\big) \geq \ell(R)^\gamma \ell(Q)^{1-\gamma}. \tag{2.10}$$

Otherwise R is said to be $\mathcal{D}_B(\omega)$-*bad*. Note that the systems $\mathcal{D}_0$ and $\mathcal{D}(w)$ depend on the ball B, but later when we use these systems it should be clear what the ball is. Also, with our definition, every cube $R \in \mathcal{D}_0$ with $\ell(R) \geq \delta^r \ell(Q_B(\omega_0))$ is automatically $\mathcal{D}_B(\omega)$-good.

A key property of these good and bad cubes is that under a random choice of $\omega \in \Omega$ a cube $R \in \mathcal{D}_0$ has a small probability of being $\mathcal{D}_B(\omega)$-bad. The version of this fact that we will use is formulated in the following lemma. For every $k, l \in \mathbb{Z}, k \leq l$, define

$$\Omega_k^l = \{\omega \in \Omega : \omega(m) = \omega_0(m) \text{ if } m < k \text{ or } m > l\}.$$

We equip Ω_k^l again with the natural probability measure with which the coordinates $\omega(m), k \leq m \leq l$, are independent and uniformly distributed over $\{0, \dots, L\} \times \{1, \dots, M\}$. Note that this is a finite probability space.

2.11. **Lemma.** *There exist two constants $C = C(L, M) > 0$ and $\eta \in (0, 1]$ so that the following holds. Fix some big enough (depending on γ) goodness parameter r. Suppose $B \subset E$ is a ball in E and let $\mathcal{D}_0 = \mathcal{D}(\omega_0)$ and $\mathcal{D}_B(\omega) \subset \mathcal{D}(\omega)$ be the dyadic lattices related to this ball as described above. Assume $k_0 \in \mathbb{Z}$ is such that $\ell(Q_B(\omega)) = \delta^{k_0}$ for some, and hence for every, $\omega \in \Omega$. Let $k_1 \in \mathbb{Z}$ be any number such that $k_1 \geq k_0 + r$. Then, for every cube $R \in \mathcal{D}_0$ with $\ell(R) \geq \delta^{k_1}$ it holds that*

$$\mathbb{P}\big(\{\omega \in \Omega_{k_0}^{k_1} : R \textit{ is } \mathcal{D}_B(\omega)\textit{-bad}\}\big) \leq C\delta^{\gamma r \eta}. \tag{2.12}$$

The point in the reduction to these finite spaces $\Omega_{k_0}^{k_1}$ is a certain technical problem related to measurability in the big piece global Tb theorem. If one replaces $\Omega_{k_0}^{k_1}$ with Ω in (2.12), then the inequality would follow from [10], Theorem 10.2. In a similar way as in [10] Inequality (2.12) is also essentially proved in [2], Theorem 2.11. In Appendix A we sketch the proof of Lemma 2.11 just by repeating arguments in [2] and noticing that it is enough to use $\Omega_{k_0}^{k_1}$ instead of the whole Ω.

3. Proof of the Main Theorem

Assuming the non-homogeneous good lambda method (Theorem 5.7) and the big pieces global Tb theorem (Theorem 6.1), we give the proof of our main theorem, Theorem 1.6, here. The following proposition is the main ingredient:

3.1. **Proposition.** *Let $C_1, C_2 \geq 1$ and $\varepsilon_0 \in (0, 1)$ be given constants. Let B be a closed ball with radius r in E. Suppose μ is a measure of order m in E and that there exists a complex measure $\nu \in M(E)$ such that*

(1) $\operatorname{spt} \nu \subset B$;
(2) $\nu(B) = \mu(B)$;
(3) $|\nu|(B) \leq C_1 \mu(B)$;
(4) If $A \subset B$ is a subset so that $\mu(A) \leq \varepsilon_0 \mu(B)$, then $|\nu|(A) \leq \frac{1}{16C_1}|\nu|(B)$.

Assume further that there exists some $s > 0$ and a Borel set $U \subset B$ so that

a) $|\nu|(U) \leq \frac{1}{16C_1}|\nu|(B)$;
b) $\sup_{\lambda > 0} \lambda^s \mu(\{y \in B \setminus U : \mathcal{C}^r \nu(y) > \lambda\}) \leq C_2 |\nu|(B)$.

Then there exists a set $G \subset B \setminus U$ with $\mu(G) > \varepsilon_0 \mu(B)$ so that

$$\|1_G \mathcal{C}_\mu f\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\mu)}$$

holds for all $f \in L^2(\mu)$ with $\{y \in E \colon f(y) \neq 0\} \subset G$.

Proof. We may assume that $\mu(E \setminus B) = 0$, because once we prove this with such measures then in the general case we may apply it with $\mu\lfloor B$.

Let $y \in B$. Then

$$\mathcal{C}_r \nu(y) \lesssim |\nu|(B)\Big(\int_{\Gamma_r(y)} d(x,E)^{-2m} \, \mathrm{d}\sigma(x)\Big)^{\frac{1}{2}} \lesssim \frac{|\nu|(B)}{r^m} \leq C_1 \frac{\mu(B)}{r^m} \lesssim 1,$$

where we used a similar estimate as in the proof of Lemma 2.4 in the second step. Hence

$$\sup_{\lambda>0} \lambda^s \mu\big(\{y \in B \setminus U \colon \mathcal{C}_r \nu(y) > \lambda\}\big) \lesssim \mu(B) \leq |\nu|(B),$$

and this combined with the weak type assumption $b)$ shows that

$$\sup_{\lambda>0} \lambda^s \mu(\{y \in B \setminus U : \mathcal{C}\nu(y) > \lambda\}) \leq C_2' |\nu|(B) \tag{3.2}$$

holds for some constant C_2'.

Let b be a function such that $|b(y)| = 1$ for all $y \in B$ and $\nu = b|\nu|$. Note that

$$\mathcal{C}\nu(y) = \mathcal{C}(b|\nu|)(y) = \mathcal{C}_{|\nu|} b(y)$$

for all $y \in E$. The idea is to use the big piece global Tb theorem 6.1 with the measure $|\nu|$ and the bounded function b. Hence we have to verify the corresponding assumptions listed in the statement of Theorem 6.1. Also, to come back to the measure μ, we will show that $|\nu|$ and μ are comparable in a big piece of B.

To begin, recall the dyadic lattices $\mathcal{D}_B(\omega)$ related to the ball B. Fix some random parameter $\omega \in \Omega$ and let $\mathcal{A}_\omega$ be the collection of the maximal dyadic cubes $R \in \mathcal{D}_B(\omega)$ such that

$$\Big|\int_R b \, \mathrm{d}|\nu|\Big| \leq \eta |\nu|(R),$$

where $\eta \in (0,1)$ is a small number to be specified. Set $T_\omega := \bigcup_{R \in \mathcal{A}_\omega} R$. First estimate

$$\int_{Q_B(\omega)} b \, \mathrm{d}|\nu| = \nu(Q_B(\omega)) = \mu(B) \geq \frac{1}{C_1} |\nu|(Q_B(\omega)).$$

Using this we get

$$\begin{aligned} \frac{1}{C_1} |\nu|(Q_B(\omega)) &\leq \int_{Q_B(\omega)} b \, \mathrm{d}|\nu| = \int_{Q_B(\omega) \setminus T_\omega} b \, \mathrm{d}|\nu| + \sum_{R \in \mathcal{A}_\omega} \int_R b \, \mathrm{d}|\nu| \\ &\leq |\nu|(Q_B(\omega) \setminus T_\omega) + \eta |\nu|(T_\omega) \\ &= |\nu|(Q_B(\omega)) + (\eta - 1)|\nu|(T_\omega), \end{aligned}$$

which can be written as

$$|\nu|(T_\omega) \leq \frac{C_1 - 1}{C_1(1-\eta)} |\nu|(Q_B(\omega)).$$

If $\eta = \eta(C_1)$ is chosen suitably, then

$$\frac{C_1 - 1}{C_1(1-\eta)} = 1 - \frac{1}{2C_1} < 1.$$

Next, define

$$H_0 := \{y \in E : M^m \nu(y) > p_0\},$$

where $p_0 > 0$ will be fixed. For every $y \in H_0$ set

$$r(y) := \sup\Big\{r > 0 : \frac{|\nu|\big(B(y,r)\big)}{r^m} > p_0\Big\}.$$

If $y \in H_0$ and $z \in B(y, r(y))$, then

$$\frac{|\nu|\big(B(z,2r(y))\big)}{(2r(y))^m} \geq 2^{-m}\frac{|\nu|\big(B(y,r(y))\big)}{r(y)^m} \geq 2^{-m}p_0.$$

Hence, if we define

$$H_1 := \bigcup_{y \in H_0} B(y, r(y)),$$

we see that $H_1 \subset \{y \in E \colon M^m\nu(y) \geq 2^{-m}p_0\}$.

Because of the weak $(1,1)$ boundedness of the maximal function and the assumption (3), we have

$$\mu(H_1) \leq \mu(\{y \in E \colon M^m\nu(y) \geq 2^{-m}p_0\}) \leq C\frac{2^m}{p_0}|\nu|(B) \leq C\frac{2^m}{p_0}C_1\mu(B).$$

Set $p_0 := C2^mC_1\varepsilon_0^{-1}$, whence the assumption (4) gives $|\nu|(H_1) \leq \frac{1}{16C_1}|\nu|(B)$.

Now we prove the comparability of the measures μ and $|\nu|$ in a subset of B. Define $\mathcal{F}_1$ to be the set of maximal cubes $R \in \mathcal{D}_B(\omega)$ such that

$$|\nu|(R) \leq \frac{1}{16C_1}\mu(R),$$

and $\mathcal{F}_2$ to be the set of maximal cubes $R \in \mathcal{D}_B(\omega)$ such that

$$|\nu|(R) \geq \frac{C_1}{\varepsilon_0}\mu(R).$$

As an immediate consequence of the definition of $\mathcal{F}_1$ we get

$$\begin{aligned} |\nu|\Big(\bigcup_{R \in \mathcal{F}_1} R\Big) = \sum_{R \in \mathcal{F}_1} |\nu|(R) &\leq \frac{1}{16C_1}\sum_{R \in \mathcal{F}_1}\mu(R) \\ &\leq \frac{1}{16C_1}\mu(Q_B(\omega)) \\ &\leq \frac{1}{16C_1}|\nu|(B). \end{aligned}$$

Also, it holds that

$$\sum_{R \in \mathcal{F}_2}\mu(R) \leq \sum_{R \in \mathcal{F}_2}\frac{\varepsilon_0}{C_1}|\nu|(R) \leq \frac{\varepsilon_0}{C_1}|\nu|(B) \leq \varepsilon_0\mu(B),$$

and accordingly

$$|\nu|\Big(\bigcup_{R\in\mathcal{F}_2} R\Big) \leq \frac{1}{16C_1}|\nu|(B)$$

by assumption (4) again. Hence the set

$$H_2 := \bigcup_{R\in\mathcal{F}_1\cup\mathcal{F}_2} R$$

satisfies

$$|\nu|(H_2) \leq \frac{1}{8C_1}|\nu|(B).$$

If $Q \in \mathcal{D}_B(\omega)$ is such that $Q \not\subset H_2$, then

$$\frac{1}{16C_1}\mu(Q) \leq |\nu|(Q) \leq \frac{C_1}{\varepsilon_0}\mu(Q). \tag{3.3}$$

From this we can conclude (using a dyadic variant of Lemma 2.13 of [16]) that for all Borel sets $A \subset \mathbb{R}^n$ there holds that

$$\frac{1}{16C_1}\mu(A\cap(B\setminus H_2)) \leq |\nu|(A\cap(B\setminus H_2)) \leq \frac{C_1}{\varepsilon_0}\mu(A\cap(B\setminus H_2)).$$

In particular, we have that $|\nu|\lfloor(B\setminus H_2) \ll \mu\lfloor(B\setminus H_2)$. Radon–Nikodym theorem gives us a Borel function $\varphi \geq 0$ so that

$$|\nu|(A) = \int_A \varphi\,\mathrm{d}\mu$$

and

$$\frac{1}{16C_1} \leq \varphi(y) \leq \frac{C_1}{\varepsilon_0} \tag{3.4}$$

hold for all Borel sets $A \subset B\setminus H_2$ and μ-a.e. $y \in B\setminus H_2$.

Now we have constructed all the necessary sets. Define $H := H_1\cup H_2\cup U$. If $|\nu|(B(y,r)) > p_0 r^m$, then $B(y,r)\subset H$, and

$$\begin{aligned}|\nu|(T_\omega\cup H) &\leq |\nu|(T_\omega) + |\nu|(H_1)\cup|\nu|(H_2)\cup|\nu|(U)\\ &\leq \Big(1-\frac{1}{2C_1}+\frac{1}{16C_1}+\frac{1}{8C_1}+\frac{1}{16C_1}\Big)|\nu|(B) = \Big(1-\frac{1}{4C_1}\Big)|\nu|(B).\end{aligned}$$

Furthermore, the weak type condition (3.2) and Equation (3.4) give

$$\begin{aligned}\sup_{\lambda>0}\lambda^s|\nu|\big(\{y\in B\setminus H\colon \mathcal{C}_{|\nu|}b>\lambda\}\big) &\lesssim \sup_{\lambda>0}\lambda^s\mu\big(\{y\in B\setminus U\colon \mathcal{C}\nu(y)>\lambda\}\big)\\ &\leq C_2'|\nu|(B).\end{aligned}$$

Since the assumptions of the big piece global Tb theorem 6.1 are verified, we may apply it to give a set $G\subset B\setminus H$ with

$$|\nu|(G) \geq \frac{1-(1-\frac{1}{4C_1})}{3}|\nu|(B) = \frac{1}{12C_1}|\nu|(B) \tag{3.5}$$

such that

$$\|1_G \mathcal{C}_{|\nu|} f\|_{L^2(|\nu|)} \lesssim_{C_1,C_2,\varepsilon_0} \|f\|_{L^2(|\nu|)}$$

holds for every $f \in L^2(|\nu|)$. Note that it must be that $\mu(G) > \varepsilon_0 \mu(B)$, because otherwise (3.5) would be contradicted by the assumption (4).

Suppose now $f \in L^2(\mu)$ with $\{y \in E \colon f(y) \neq 0\} \subset G$ and define the function $g := f/\varphi$, which is understood to be zero in $\{y \in E \colon f(y) = 0\}$. Remember that $|\nu|\lfloor(B \setminus H_2) = \varphi\mu\lfloor(B \setminus H_2)$ and $\varphi \sim 1$ for $|\nu|$- and μ- a.e. $y \in B \setminus H$. Then

$$\begin{aligned}\|1_G \mathcal{C}_\mu f\|_{L^2(\mu)} = \|1_G \mathcal{C}_{|\nu|} g\|_{L^2(\mu)} \sim \|1_G \mathcal{C}_{|\nu|} g\|_{L^2(|\nu|)} &\lesssim \|g\|_{L^2(|\nu|)} \\ &= \|f/\varphi\|_{L^2(|\nu|)} \\ &\sim \|f\|_{L^2(\mu)}.\end{aligned}$$

This concludes the proof. □

With this proposition we can easily prove the main theorem:

Proof of Theorem 1.6. Let B be a closed $(10, b)$-doubling ball in E with a κ-small boundary. Then there exists by assumption a measure ν_B related to the ball B as in Proposition 3.1. Thus, an application of that proposition gives a set $G_B \subset B$ with $\mu(G_B) > \varepsilon_0 \mu(B)$ such that

$$\|1_{G_B} \mathcal{C}_\mu f\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\mu)}$$

holds for all $f \in L^2(\mu)$ with $\{y \in E \colon f(y) \neq 0\} \subset G_B$.

Since this happens in every closed $(10, b)$-doubling ball with a κ-small boundary, the good lambda theorem 5.7 and Remark 5.8 following it imply that $\mathcal{C}_\mu$ is bounded in $L^p(\mu)$ for every $p \in (1, \infty)$. □

4. A GEOMETRIC PROBLEM RELATED TO CONES

Before going to the good lambda method and the big pieces global Tb theorem, we consider a certain geometric problem related to cones. Namely, we want to estimate the σ-measure of two truncated cones that are close to each other. Compared to the upper half-space case, the difficulty here is that a cone defined with respect to a general set E does not have such a simple form. This estimate will be needed to have certain continuity for a truncated square function.

4.1. **Lemma.** *Let $t \geq 10$. Then for all $r > 0$ and $y, y' \in E$ with $|y - y'| < r$ it holds that*

$$\sigma\big(\Gamma_{tr}(y) \Delta \Gamma_{tr}(y')\big) \lesssim \frac{1}{t}, \tag{4.2}$$

where $\Gamma_{tr}(y)\Delta\Gamma_{tr}(y') := \Gamma_{tr}(y) \setminus \Gamma_{tr}(y') \cup \Gamma_{tr}(y') \setminus \Gamma_{tr}(y)$.

Proof. Let $x \in \Gamma_{tr}(y) \setminus \Gamma_{tr}(y')$. The crucial observation is that then

$$|x - y| < 2d(x, E) \leq |x - y'| < |x - y| + r, \tag{4.3}$$

so x is at a quite specific distance to E. Hence there exists $\tilde{y}_x \in E$ so that

$$|x - \tilde{y}_x| < \frac{|x-y| + r}{2},$$

and for all $\tilde{y} \in E$ there holds that

$$|x - \tilde{y}| > \frac{|x-y|}{2}.$$

For every $\tilde{y} \in E$ define the sets

$$\bar{B}_{\tilde{y}} := \Big\{ x \in \mathbb{R}^n : |x - \tilde{y}| \le \frac{|x-y|}{2} \Big\} \tag{4.4}$$

($\bar{B}$ indicates that it will turn out to be a closed ball) and

$$G_{\tilde{y}} := \Big\{ x \in \mathbb{R}^n : |x - \tilde{y}| < \frac{|x-y| + r}{2} \Big\}. \tag{4.5}$$

The above considerations show that

$$\Gamma_{tr}(y) \setminus \Gamma_{tr}(y') \subset \Big[\bigcup_{\tilde{y} \in E} G_{\tilde{y}} \setminus \bigcup_{\tilde{y} \in E} \bar{B}_{\tilde{y}} \Big] \cap \Gamma_{tr}(y).$$

Since every $G_{\tilde{y}}$ is open, there is a countable collection $\{y_i\}_{i \in \tilde{\mathcal{I}}} \subset E$ so that $\bigcup_{i \in \tilde{\mathcal{I}}} G_{y_i} = \bigcup_{\tilde{y} \in E} G_{\tilde{y}}$. Then, we can find a finite subcollection $\mathcal{I} \subset \tilde{\mathcal{I}}$ so that

$$\begin{aligned} \sigma\Big(\Big[\bigcup_{\tilde{y} \in E} G_{\tilde{y}} \setminus \bigcup_{\tilde{y} \in E} \bar{B}_{\tilde{y}} \Big] \cap \Gamma_{tr}(y) \Big) &\le 2\sigma\Big(\Big[\bigcup_{i \in \mathcal{I}} G_{y_i} \setminus \bigcup_{\tilde{y} \in E} \bar{B}_{\tilde{y}} \Big] \cap \Gamma_{tr}(y) \Big) \\ &\le 2\sigma\Big(\Big[\bigcup_{i \in \mathcal{I}} G_{y_i} \setminus \bigcup_{i \in \mathcal{I}} \bar{B}_{y_i} \Big] \cap \Gamma_{tr}(y) \Big). \end{aligned}$$

We can clearly assume that $y_i \neq y_j$, for all $i, j \in \mathcal{I}, i \neq j$. For every $k \in \{0, 1, 2, \ldots\}$ let $\mathcal{I}_k \subset \mathcal{I}$ be the set of those indices i such that $G_{y_i} \cap \Gamma_{2^k tr}^{2^{k+1} tr}(y) \neq \emptyset$, whence

$$\sigma\Big(\Big[\bigcup_{i \in \mathcal{I}} G_{y_i} \setminus \bigcup_{i \in \mathcal{I}} \bar{B}_{y_i} \Big] \cap \Gamma_{tr}(y) \Big) = \sum_{k=0}^{\infty} \sigma\Big(\Big[\bigcup_{i \in \mathcal{I}_k} G_{y_i} \setminus \bigcup_{i \in \mathcal{I}_k} \bar{B}_{y_i} \Big] \cap \Gamma_{2^k tr}^{2^{k+1} tr}(y) \Big). \tag{4.6}$$

Now we fix some k for the rest of the proof and show that

$$m_n\Big(\bigcup_{i \in \mathcal{I}_k} G_{y_i} \setminus \bigcup_{i \in \mathcal{I}_k} \bar{B}_{y_i} \Big) \lesssim (2^k tr)^{n-1} r, \tag{4.7}$$

where m_n is the n-dimensional Lebesgue measure. Using this we may infer from (4.6) that

$$\sigma\Big(\Big[\bigcup_{i \in \mathcal{I}} G_{y_i} \setminus \bigcup_{i \in \mathcal{I}} \bar{B}_{y_i} \Big] \cap \Gamma_{tr}(y) \Big) \lesssim \sum_{k=0}^{\infty} \frac{(2^k tr)^{n-1} r}{(2^k tr)^n} \sim \frac{1}{t},$$

which proves the lemma.

$\bar{B}_{y_i}$ **is a ball.** Let $i \in \mathcal{I}_k$. First we'll show that $\bar{B}_{y_i}$ is a closed ball. We write a general point $x \in \mathbb{R}^n$ with coordinates as $x = (x(1), \dots, x(n))$. The condition $|x - y_i| \le \frac{|x-y|}{2}$ can be written as

$$\sum_{m=1}^{n} \big(x(m) - y_i(m)\big)^2 \le \frac{1}{4} \sum_{m=1}^{n} \big(x(m) - y(m)\big)^2,$$

and further as

$$\sum_{m=1}^{n} \big(x(m) - (\frac{4}{3}y_i(m) - \frac{1}{3}y(m))\big)^2 \le \frac{4}{9} \sum_{m=1}^{n} \big(y(m) - y_i(m)\big)^2.$$

From here we see that

$$\bar{B}_{y_i} = \bar{B}\big(\frac{4}{3}y_i - \frac{1}{3}y, \frac{2}{3}|y - y_i|\big). \tag{4.8}$$

The centers of the balls $\bar{B}_{y_i}$ and $\bar{B}_{y_j}$ are different if $i \ne j$.

Consider still the fixed $i \in \mathcal{I}_k$. By definition there exists a point $x \in G_{y_i} \cap \Gamma_{2^k tr}^{2^{k+1}tr}(y)$. The distance $|y - y_i|$ can be estimated as

$$|y - y_i| \le |y - x| + |x - y_i| < |y - x| + \frac{|y - x| + r}{2} \le 2|x - y|,$$

because $|y - x| > 10r$. On the other hand

$$|y - y_i| \ge |x - y| - |x - y_i| > |x - y| - \frac{|x - y| + r}{2} \ge \frac{|x - y|}{3}.$$

Combining these we get

$$r(\bar{B}_{y_i}) = \frac{2}{3}|y - y_i| \sim |x - y| \sim 2^k tr. \tag{4.9}$$

Also, from (4.9) and the fact

$$\Big|\Big(\frac{4}{3}y_i - \frac{1}{3}y\Big) - y\Big| = \frac{4}{3}|y - y_i| \sim 2^k tr$$

it follows that there exists an absolute constant C such that

$$\bar{B}_{y_i} \subset B(y, C2^k tr). \tag{4.10}$$

. Re-index the balls $\{\bar{B}_{y_i}\}_{i \in \mathcal{I}_k}$ as $\bar{B}_1 = \bar{B}(x_1, r_1), \dots, \bar{B}_K = \bar{B}(x_K, r_K)$ for some $K \in \mathbb{N}$, and write correspondingly $G_1, \dots, G_K$. Then, set

$$A_1 := \partial\Big(\bigcup_{j=1}^{K} \bar{B}_j\Big) \cap \partial \bar{B}_1,$$

and for every $i \in \{2, \dots, K\}$ define

$$A_i := \Big\{x \in \partial\Big(\bigcup_{j=1}^{K} \bar{B}_j\Big) : x \in \partial \bar{B}_i \setminus \Big(\bigcup_{j=1}^{i-1} \partial \bar{B}_j\Big)\Big\}.$$

The sets A_i are pairwise disjoint and $\bigcup_{i=1}^K A_i = \partial\Big(\bigcup_{j=1}^K \bar{B}_j\Big)$. With the sets A_i we still define

$$V_i = \big\{x \in \mathbb{R}^n : x = (1-\lambda)x_i + \lambda a \text{ for some } a \in A_i \text{ and } \lambda \in [0,1]\big\}.$$

In other words, V_i is the set of points that are on a segment whose one end is x_i and the other is on A_i.

The sets V_i are pairwise disjoint. We claim that $V_i \cap V_j = \emptyset$ if $i \neq j$. To get a contradiction, suppose that there exist $a_i \in A_i$, $a_j \in A_j$ and λ_i, $\lambda_j \in [0,1]$ such that

$$x = (1-\lambda_i)x_i + \lambda_i a_i = (1-\lambda_j)x_j + \lambda_j a_j, \tag{4.11}$$

where $i, j \in \{1, \ldots, K\}$ and $i \neq j$. Assume first $|x - a_j| > |x - a_i|$ and notice that by (4.11),

$$a_i \in B(x, |x-a_j|) \subset B(x_j, |x_j - a_j|) \subset \text{int}\Big(\bigcup_{l=1}^K \bar{B}_l\Big),$$

which is a contradiction because a_i is supposed to be on the boundary. The case $|x - a_i| > |x - a_j|$ is handled similarly. Thus, we can only have that

$$|x - a_i| = |x - a_j|. \tag{4.12}$$

Without loss of generality assume that

$$|x_i - a_i| \geq |x_j - a_j|. \tag{4.13}$$

Suppose first $\lambda_i \neq 0$. Then

$$a_j \in \bar{B}(x, |x - a_j|) = \bar{B}(x, |x-a_i|) \subset B(x_i, |x_i - a_i|) \cup \{a_i\},$$

which is a contradiction. Indeed, we have $a_j \notin B(x_i, |x_i - a_i|)$ because a_j is a boundary point, and $a_i \neq a_j$ since the sets A_i are pairwise disjoint.

Suppose then $\lambda_i = 0$, which implies that $x = x_i$. This combined with (4.11), (4.12) and (4.13) implies that $\lambda_j = 0$, since

$$|x - a_j| = |x - a_i| = |x_i - a_i| \geq |x_j - a_j|.$$

This gives $x_j = x = x_i$. This is again a contradiction because $x_i \neq x_j$ (as noted after (4.8)). Thus we have shown that $V_i \cap V_j = \emptyset$ if $i \neq j$.

Proof of (4.7)**.** Let $x \in \bigcup_{j=1}^K G_j \setminus \bigcup_{j=1}^K \bar{B}_j$, and suppose $i \in \{1, \ldots, K\}$ and $a \in A_i$ are such that

$$|x - a| = d\Big(x, \bigcup_{j=1}^K A_j\Big) = d\Big(x, \bigcup_{j=1}^K \bar{B}_j\Big). \tag{4.14}$$

Then, because a minimizes the distance of x to the ball $\bar{B}(x_i, r_i)$, x has to be on the same line with a and x_i. Otherwise there would be a point $x' \in \bar{B}_i$ with

$|x - x'| < |x - a|$, which contradicts (4.14). From the definitions (4.4) and (4.5) of the sets $\bar{B}_i$ and G_i it follows that $|x - a| \le r$. Thus

$$\bigcup_{i=1}^{K} G_i \setminus \bigcup_{i=1}^{K} \bar{B}_i \subset \bigcup_{i=1}^{K} F_i, \tag{4.15}$$

where

$$F_i := \big\{x_i + t(a - x_i) \colon t \in \big(1, 1 + r/r_i\big], a \in A_i\big\}.$$

Fix some i for the moment and recall the set V_i from above. We want to compare the Lebesgue measures of F_i and V_i. Note that the set $(V_i - x_i) \setminus \{\bar{0}\}$ can be written as a disjoint union

$$(V_i - x_i) \setminus \{\bar{0}\} = \bigcup_{k=1}^{\infty} \Big(\frac{r_i}{r_i + r}\Big)^k (F_i - x_i).$$

Hence

$$m_n(V_i) = \sum_{k=1}^{\infty} \Big(\frac{r_i}{r_i + r}\Big)^{nk} m_n(F_i) = \frac{r_i^n}{(r_i + r)^n - r_i^n} m_n(F_i).$$

Using the mean value theorem and the fact that $r_i \sim 2^k tr$ we have

$$\frac{r_i^n}{(r_i + r)^n - r_i^n} \sim \frac{r_i}{r} \sim 2^k t,$$

and thus $m_n(V_i) \sim 2^k t \cdot m_n(F_i)$.

Remember that the sets $V_i \subset \bar{B}_i$ are pairwise disjoint and that $\bar{B}_i \subset B(y, C2^k tr)$ for every $i \in \{1, \dots, K\}$, as stated in (4.10). Now we can estimate

$$\sum_{i=1}^{K} m_n(F_i) \sim \frac{1}{2^k t} \sum_{i=1}^{K} m_n(V_i) \le \frac{m_n(B(y, C2^k tr))}{2^k t} \sim (2^k tr)^{n-1} r,$$

which in view of (4.15) completes the proof of (4.7). The proof of Lemma 4.1 is complete.

□

5. The non-homogeneous good lambda method

In this section we prove the non-homogeneous good lambda method of Tolsa [20] in our setting. For this, we shall need the geometric considerations from Section 4.

In the proof of the good lambda inequality we shall use the following Whitney type argument, Lemma 5.2, which allows the use of regular balls only. This is a version of Lemma 2.23 in [20] adapted to our situation, with some additional arguments from [14] related to small boundaries and the usage of balls.

First, we record the following fact from [17] (see also [20] and [21]).

5.1. **Lemma.** *Let μ be a Radon measure in $\mathbb{R}^n$ and let κ be a big enough constant depending only on the dimension n. Suppose $B(x,r)$ is a ball (open or closed) in $\mathbb{R}^n$. Then there exists $R \in [r, 1.2r]$ so that the ball $B(x,R)$ has a κ-small boundary.*

If $B = B(x,r)$ is a ball in E or in $\mathbb{R}^n$ and $s > 0$, we define $sB := sB(x,r) := B(x,sr)$, and similarly with closed balls.

5.2. **Lemma.** *Let μ be a Borel measure with* $\operatorname{spt}\mu \subset E$. *Suppose $U \subsetneq E$ is a relatively open set with $\mu(U) < \infty$. Assume $a \geq 3, \rho \geq 16a$ and let b be a big enough constant depending on ρ and the dimension n. Let also κ be a big enough constant depending only on the dimension n. Recall the constant $\delta = \frac{1}{1000}$ related to dyadic lattices in E. Define $C_1 := \frac{\rho}{8} \geq 6$ and $C_2 := \frac{(12+\rho)\delta^{-1}}{6} \geq 10000$. Then there exist a constant $D_0 = D_0(\rho, n)$ and a finite collection of closed balls $\{B_i\}_{i\in\mathcal{I}}$* in *$E$ with the following properties:*

- *$B_i \cap B_j = \emptyset$ if $i \neq j$.*
- *For every $i \in \mathcal{I}$ there exist at most D_0 indices $j \in \mathcal{I}$ so that $\frac{C_1}{2}B_i \cap \frac{C_1}{2}B_j \neq \emptyset$.*
- *For every $i \in \mathcal{I}$ it holds that $C_1 B_i \subset U$ and $C_2 B_i \cap (E \setminus U) \neq \emptyset$.*
- *The balls B_i are (a,b)-doubling and have κ-small boundary.*
- *$\mu\big(\bigcup_{i\in\mathcal{I}} B_i\big) \geq \frac{1}{2b}\mu(U)$.*

Proof. Let $\mathcal{D}$ be any dyadic lattice in E as described in Section 2. Consider the maximal dyadic cubes $Q \in \mathcal{D}$ such that

$$d(Q, E \setminus U) \geq \rho\ell(Q). \tag{5.3}$$

That Q is a maximal cube such that (5.3) holds means that there does not exist a cube $R \in \mathcal{D}$ satisfying (5.3) so that $R \supset Q$ and $\ell(R) > \ell(Q)$. Let $\{Q_i\}_{i\in\mathcal{K}} \subset \mathcal{D}$ be the collection of these maximal cubes. Then $U = \bigcup_{i\in\mathcal{K}} Q_i$ and the cubes in $\{Q_i\}_{i\in\mathcal{K}}$ are pairwise disjoint.

Suppose $i \in \mathcal{K}$. Recall that if $Q \in \mathcal{D}$ then $\widehat{Q} \in \mathcal{D}$ is the unique cube with $\ell(\widehat{Q}) = \delta^{-1}\ell(Q)$ that contains Q. By construction we know that $d(\widehat{Q_i}, E \setminus U) < \rho\ell(\widehat{Q_i})$. Hence

$$\begin{aligned} d(c_{Q_i}, E \setminus U) &\leq \operatorname{diam}(\widehat{Q_i}) + d(\widehat{Q_i}, E \setminus U) \\ &< 12\ell(\widehat{Q_i}) + \rho\ell(\widehat{Q_i}) \\ &= (12+\rho)\delta^{-1}\ell(Q_i) = C(\rho)\ell(Q_i), \end{aligned}$$

where $C(\rho) := (12+\rho)\delta^{-1}$. Hence, it holds for all $i \in \mathcal{K}$ that

$$\rho\ell(Q_i) \leq d(c_{Q_i}, E \setminus U) < C(\rho)\ell(Q_i). \tag{5.4}$$

Next we prove the existence of the constant D_0. Suppose $i, j \in \mathcal{K}$ so that

$$\bar{B}_E(c_{Q_i}, \frac{\rho}{2}\ell(Q_i)) \cap \bar{B}_E(c_{Q_j}, \frac{\rho}{2}\ell(Q_j)) \neq \emptyset,$$

and suppose $\ell(Q_j) = \delta^k \ell(Q_i)$ for some $k \in \mathbb{Z}, k \geq 0$. Then

$$\begin{aligned} d(c_{Q_i}, E \setminus U) &\leq |c_{Q_i} - c_{Q_j}| + d(c_{Q_j}, E \setminus U) \\ &\leq \frac{\rho}{2}\ell(Q_i) + \frac{\rho}{2}\ell(Q_j) + C(\rho)\ell(Q_j) \\ &= (\frac{\rho}{2} + \frac{\rho}{2}\delta^k + C(\rho)\delta^k)\ell(Q_i). \end{aligned}$$

Thus, because of (5.4), we see that there exists $k_0 \in \mathbb{Z}$ depending on ρ such that $k \leq k_0$.

Fix now some ball $\bar{B}_E(c_{Q_i}, \frac{\rho}{2}\ell(Q_i))$ and let $\mathcal{K}_i$ be the set of those indices j such that $\bar{B}_E(c_{Q_i}, \frac{\rho}{2}\ell(Q_i)) \cap \bar{B}_E(c_{Q_j}, \frac{\rho}{2}\ell(Q_j)) \neq \emptyset$. Then for all $j \in \mathcal{K}_i$ it holds that

$$\delta^{k_0}\ell(Q_j) \leq \ell(Q_i) \leq \delta^{-k_0}\ell(Q_j).$$

Hence

$$c_{Q_j} \in \bar{B}_E\big(c_{Q_i}, \frac{\rho}{2}(1 + \delta^{-k_0})\ell(Q_i)\big) \quad \text{for all } j \in \mathcal{K}_i. \tag{5.5}$$

Also, if $j, j' \in \mathcal{K}_i, j \neq j'$, then because the cubes $Q_i, i \in \mathcal{K}$, are pairwise disjoint, we have

$$|c_{Q_j} - c_{Q_{j'}}| \geq \max\big(\frac{1}{6}\ell(Q_j), \frac{1}{6}\ell(Q_{j'})\big) \geq \frac{1}{6}\delta^{k_0}\ell(Q_i). \tag{5.6}$$

Equations (5.5) and (5.6) combined imply that the number of indices in $\mathcal{K}_i$ is bounded by a constant D_0 that depends only on ρ and n.

. Now we start forming the collection we are after. Suppose $i \in \mathcal{K}$ and consider the ball $\bar{B}_E(c_{Q_i}, 6\ell(Q_i))$. Let $B_i := \bar{B}_E(c_{Q_i}, r_i)$ be a ball with a κ-small boundary and radius $r_i \in [6\ell(Q_i), 1.2 \cdot 6\ell(Q_i)]$ given by Lemma 5.1. Since $Q_i \subset B_i \subset U$ we have

$$U = \bigcup_{i \in \mathcal{K}} B_i.$$

Also, since $1.2 \cdot 6 \cdot \frac{C_1}{2} = 1.2 \cdot 6 \cdot \frac{\rho}{16} \leq \frac{\rho}{2}$, for every $i \in \mathcal{I}$ there exist at most D_0 indices $j \in \mathcal{I}$ so that $\frac{C_1}{2}B_i \cap \frac{C_1}{2}B_j \neq \emptyset$.

Let $\mathcal{S} \subset \mathcal{K}$ be the set of indices such that the balls B_i are (a, b)-doubling with respect to μ. Then, since $1.2 \cdot 6a \leq \frac{\rho}{2}$, we have

$$\begin{aligned} \mu(\bigcup_{i \in \mathcal{K} \setminus \mathcal{S}} B_i) &\leq \sum_{i \in \mathcal{K} \setminus \mathcal{S}} \mu(B_i) \\ &\leq b^{-1} \sum_{i \in \mathcal{I} \setminus \mathcal{S}} \mu(aB_i) \\ &\leq \frac{D_0}{b}\mu(U). \end{aligned}$$

So, if b is big enough, then

$$\mu\Big(\bigcup_{i \in \mathcal{S}} B_i\Big) \geq \frac{2}{3}\mu(U),$$

and choosing a sufficiently big finite subcolletion $\mathcal{S}_1 \subset \mathcal{S}$, we get

$$\mu\Big(\bigcup_{i\in\mathcal{S}_1} B_i\Big) \geq \frac{1}{2}\mu(U).$$

Finally, using the $3r$-covering theorem, choose a subcollection $\mathcal{I} \subset \mathcal{S}_1$ so that the balls $B_i, i \in \mathcal{I}$, are pairwise disjoint and

$$\bigcup_{i\in\mathcal{S}_1} B_i \subset \bigcup_{i\in\mathcal{I}} 3B_i.$$

Then, since $a \geq 3$, we have

$$\begin{aligned}\sum_{i\in\mathcal{I}} \mu(B_i) &\geq b^{-1}\sum_{i\in\mathcal{I}} \mu(3B_i)\\ &\geq b^{-1}\mu\Big(\bigcup_{i\in\mathcal{S}_1} B_i\Big)\\ &\geq \frac{1}{2b}\mu(U).\end{aligned}$$

The collection $\{B_i\}_{i\in\mathcal{I}}$ satisfies all the desired properties.

□

5.7. **Theorem.** *Let μ be a measure of order m in E. Let also $b, \kappa > 0$ be big enough constants depending only on n, and assume $\theta \in (0,1)$. Suppose for each closed $(10,b)$-doubling ball B in E with a κ-small boundary there exists a subset $G_B \subset B$ with $\mu(G_B) \geq \theta\mu(B)$ so that $\mathcal{C}\colon \mathcal{M}(E) \to L^{1,\infty}(\mu\lfloor G_B)$ is bounded with a uniform constant independent of B. Then $\mathcal{C}_\mu$ is bounded in $L^p(\mu)$ for all $p \in (1,\infty)$ with a constant depending on p and the preceding constants.*

5.8. *Remark.* In Theorem 5.7 the assumption that $\mathcal{C}\colon \mathcal{M}(E) \to L^{1,\infty}(\mu\lfloor G_B)$ is bounded can be replaced by the assumption that $\mathcal{C}_\mu\colon L^2(\mu\lfloor G_B) \to L^2(\mu\lfloor G_B)$ is bounded, because the latter implies the former using standard reasoning. This is proved in Appendix B.

Proof of Theorem 5.7. Fix an exponent $p \in (1,\infty)$. Since for any $f \in L^p(\mu)$ it holds that

$$\mathcal{C}_{\mu,s}^t f(y) = \Big(\int_{\Gamma_s^t(y)} |T_\mu f(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\Big)^{\frac{1}{2}} \nearrow \mathcal{C}_\mu f(y),$$

as $s \to 0$ and $t \to \infty$, it is enough to bound the operators $\mathcal{C}_{\mu,s}^t$ uniformly for $s \in (0,1)$ and $t > 1$.

Note that every $\mathcal{C}_{\mu,s}^t$ is *a priori* bounded in $L^p(\mu)$ by Lemma 2.5. Hence, it suffices to prove that

$$\|\mathcal{C}_{\mu,s}^t f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}$$

holds uniformly for bounded and boundedly supported functions f. From now on such a function f is fixed.

Note that the mapping

$$y \mapsto \mathcal{C}^t_{\mu,s} f(y), \quad y \in E,$$

is continuous, and hence the sets $\{\mathcal{C}^t_{\mu,s} f > \lambda\}$ are open in E for every $\lambda > 0$. Indeed, if $y, y' \in E$, then

$$\begin{aligned} &|\mathcal{C}^t_{\mu,s} f(y) - \mathcal{C}^t_{\mu,s} f(y')| \\ &\le \Big(\int_{\mathbb{R}^n \setminus E} |1_{\Gamma^t_s(y)} T_\mu f(x) - 1_{\Gamma^t_s(y')} T_\mu f(x)|^2 d(x,E)^{2\alpha} \,\mathrm{d}\sigma(x) \Big)^{\frac{1}{2}} \\ &\lesssim \frac{\|f\|_{L^1(\mu)}}{s^m} \sigma\big(\Gamma^t_s(y) \Delta \Gamma^t_s(y')\big)^{\frac{1}{2}}. \end{aligned} \tag{5.9}$$

When $|y - y'|$ is so small that $10|y-y'| < s$, then from (4.2) it follows that

$$\sigma\big(\Gamma^t_s(y) \Delta \Gamma^t_s(y')\big) \le \sigma\big(\Gamma_s(y) \Delta \Gamma_s(y')\big) \lesssim \frac{|y-y'|}{s},$$

which converges to zero as $y' \to y$.

The main thing to prove in the good lambda method is the *good lambda inequality* (5.10). It says that for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. If $\lambda > 0$, then

$$\mu\big(\{y \in E : \mathcal{C}^t_{\mu,s} f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \le \delta\lambda\}\big) \le \Big(1 - \frac{\theta}{4b}\Big) \mu\big(\{\mathcal{C}^t_{s,\mu} f > \lambda\}\big). \tag{5.10}$$

That $\mathcal{C}^t_{\mu,s}$ is bounded follows from this inequality and the boundedness of M_μ in a standard manner.

Fix $\lambda, \varepsilon > 0$ and let $\delta > 0$ be some number to be specified during the proof. We would like to begin the proof of (5.10) by applying the Whitney type lemma 5.2 to the relatively open set $\{\mathcal{C}^t_{\mu,s} f > \lambda\} =: \Omega_\lambda$, and for this reason we need that $\Omega_\lambda \subsetneq E$.

Suppose $\operatorname{diam}(E) = \infty$. If $y \notin \operatorname{spt} f$, then

$$\begin{aligned} \mathcal{C}^t_{\mu,s} f(y)^2 = \int_{\Gamma^t_s(y)} |T_\mu f(x)|^2 d(x,E)^{2\alpha} \,\mathrm{d}\sigma(x) &\lesssim \frac{\|f\|^2_{L^1(\mu)}}{d(x, \operatorname{spt} f)^{2m}} \sigma\big(\Gamma^t_s(y)\big) \\ &\lesssim_{s,t} \frac{\|f\|^2_{L^1(\mu)}}{d(y, \operatorname{spt} f)^{2m}} \to 0, \end{aligned}$$

as $d(y, \operatorname{spt} f) \to \infty$. Hence in this case $\{\mathcal{C}^t_{\mu,s} f > \lambda\} \subsetneq E$ holds. And actually we see that Ω_λ is always a bounded set.

The case $\operatorname{diam}(E) < \infty$. Suppose $\operatorname{diam}(E) < \infty$, whence $\mu(E) < \infty$, and assume also $\Omega_\lambda = E$. To have something to prove in (5.10), we may suppose that the left hand side there is non-zero. Then there exists $y_0 \in E$ such that $M_\mu f(y_0) \le \delta\lambda$. By assumption (by the same interpretation as in Remark 1.7) there exists a set

$G_E \subset E$ with $\mu(G_E) \geq \theta\mu(E)$ where $\mathcal{C}^t_{\mu,s}\colon \mathcal{M}(E) \to L^{1,\infty}(\mu\lfloor G_E)$ is bounded. Thus

$$\begin{aligned}
\mu\big(\{y \in E : \mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \leq \delta\lambda\}\big)& \\
&\leq \mu\big(E \setminus G_E\big) + \mu\big(\{y \in G_E : \mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda\}\big) \\
&\leq (1-\theta)\mu(E) + \frac{C}{(1+\varepsilon)\lambda}\|f\|_{L^1(\mu)} \\
&= (1-\theta)\mu(E) + \frac{C\mu(E)}{(1+\varepsilon)\lambda}\frac{\|f\|_{L^1(\mu)}}{\mu(E)} \\
&\leq (1-\theta)\mu(E) + \frac{C\mu(E)}{(1+\varepsilon)\lambda}\delta\lambda \\
&\leq \Big(1-\frac{\theta}{4b}\Big)\mu(E) = \Big(1-\frac{\theta}{4b}\Big)\mu(\Omega_\lambda)
\end{aligned}$$

if $\delta > 0$ is small enough.

We have shown that if $\operatorname{diam} E < \infty$, then Inequality (5.10) holds for those λ such that $\Omega_\lambda = E$.

. So in any case we may assume that $\Omega_\lambda \subsetneq E$. As we noted above Ω_λ is a bounded set, and thus $\mu(\Omega_\lambda)$ is finite. Apply Lemma 5.2 with parameters $a = 10, \rho = 160, b$ and κ to the open set Ω_λ. This choice gives $C_1 = 20$ and $C_2 = \frac{86000}{3}$. Let $\{B_i\}_{i\in\mathcal{I}}$ be the resulting set of balls in E, and let r_i be the radius of B_i.

Since the balls B_i are closed, $(10, b)$-doubling and have a κ-small boundary, there exists for every i a set $G_i \subset B_i$ as in the assumptions. Hence

(5.11)
$$\begin{aligned}
&\mu\big(\{y \in E\colon \mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \leq \delta\lambda\}\big) \\
&\quad\leq \mu\big(\Omega_\lambda \setminus \bigcup_{i\in\mathcal{I}} B_i\big) + \sum_{i\in\mathcal{I}} \mu\big(B_i \setminus G_i\big) \\
&\quad+ \sum_{i\in\mathcal{I}} \mu\big(\{y \in G_i : \mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \leq \delta\lambda\}\big) \\
&\quad\leq \Big(1-\frac{\theta}{2b}\Big)\mu(\Omega_\lambda) + \sum_{i\in\mathcal{I}} \mu\big(\{y \in G_i : \mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \leq \delta\lambda\}\big).
\end{aligned}$$

It remains to consider the last sum above.

Step I. Fix some $i \in \mathcal{I}$. Suppose $y \in G_i$ is such that $\mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda$ and $M_\mu f(y) \leq \delta\lambda$. First we will show that then

$$\mathcal{C}^t_{\mu,s}(1_{2B_i}f)(y) > \frac{\varepsilon}{2}\lambda$$

if $\delta(\varepsilon)$ is small enough. Since

$$\mathcal{C}^t_{\mu,s}(1_{2B_i}f)(y) \geq \mathcal{C}^t_{\mu,s}f(y) - \mathcal{C}^t_{\mu,s}(1_{(2B_i)^c}f)(y),$$

this follows from showing that

$$\mathcal{C}^t_{\mu,s}(1_{(2B_i)^c} f)(y) \le \Big(1 + \frac{\varepsilon}{2}\Big)\lambda. \tag{5.12}$$

Assume for the moment that $s \le 20C_2 r_i \le t$. For $x \in \Gamma(y)$ we have by Lemma (2.2) that

$$\begin{aligned}
|T_\mu(1_{(2B_i)^c} f)(x)| &\lesssim \int_{E\setminus 2Bi} \frac{|f(z)|}{|x-z|^{m+\alpha}}\,\mathrm{d}\mu(z) \lesssim \int_{E\setminus 2B_i} \frac{|f(z)|}{|y-z|^{m+\alpha}}\,\mathrm{d}\mu(z) \\
&\lesssim \int_E \frac{|f(z)|}{(r_i+|y-z|)^{m+\alpha}}\,\mathrm{d}\mu(z) \\
&\lesssim r_i^{-\alpha} M_\mu f(y) \le r_i^{-\alpha}\delta\lambda.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{C}^{20C_2 r_i}_{\mu,s}(1_{(2B_i)^c} f)(y) &\lesssim r_i^{-\alpha}\delta\lambda\Big(\int_{\Gamma^{20C_2 r_i}(y)} d(x,E)^{2\alpha} d\sigma(x)\Big)^{\frac{1}{2}} \\
&\lesssim r_i^{-\alpha}\delta\lambda(20C_2 r_i)^\alpha \sim \delta\lambda,
\end{aligned} \tag{5.13}$$

where in the second step we used a similar estimate as in the proof of Lemma 2.4.

Because of the Whitney properties of the balls B_i there exists a point $y' \in C_2B_i \cap E \setminus \Omega_\lambda$, whence by definition $\mathcal{C}^t_{\mu,s} f(y') \le \lambda$. Thus, we can estimate

$$\begin{aligned}
\mathcal{C}^t_{\mu,s}(1_{(2B_i)^c} f)(y) &\le \mathcal{C}^{20C_2 r_i}_{\mu,s}(1_{(2B_i)^c} f)(y) + \mathcal{C}^t_{\mu,20C_2 r_i}(1_{(2B_i)^c} f)(y) \\
&\le C\delta\lambda + \big|\mathcal{C}^t_{\mu,20C_2 r_i}(1_{(2B_i)^c} f)(y) - \mathcal{C}^t_{\mu,20C_2 r_i}(1_{(2B_i)^c} f)(y')\big| \\
&\quad + \mathcal{C}^t_{\mu,20C_2 r_i}(1_{(2B_i)^c} f)(y'),
\end{aligned} \tag{5.14}$$

where further

$$\begin{aligned}
\mathcal{C}^t_{\mu,20C_2 r_i}(1_{(2B_i)^c} f)(y') &\le \mathcal{C}^t_{\mu,s} f(y') + \mathcal{C}^t_{\mu,20C_2 r_i}(1_{2B_i} f)(y') \\
&\le \lambda + \mathcal{C}^t_{\mu,20C_2 r_i}(1_{2B_i} f)(y').
\end{aligned} \tag{5.15}$$

We continue with estimating the difference in (5.14). Suppose $x \in \Gamma_{20C_2 r_i}(y')$. Then because $|y-y'| < 20^{-1} d(x,E)$, every $z \in E$ satisfies

$$|x-z| \sim d(x,E) + |y'-z| \sim d(x,E) + |y-z|.$$

This gives $|T_\mu(1_{(2B_i)^c} f)(x)| \lesssim d(x,E)^{-\alpha} M_\mu f(y)$ by Lemma 2.1. Since the same estimate holds for all $x \in \Gamma(y)$, we have

$$\begin{aligned}
&|\mathcal{C}^t_{\mu,20C_2 r_i}(1_{(2B_i)^c} f)(y) - \mathcal{C}^t_{\mu,20C_2 r_i}(1_{(2B_i)^c} f)(y')| \\
&\qquad \le \Big(\int_{\Gamma_{20C_2 r_i}(y)\Delta\Gamma_{20C_2 r_i}(y')} |T_\mu(1_{(2B_i)^c} f)(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\Big)^{\frac{1}{2}} \\
&\qquad \lesssim \delta\lambda\sigma\Big(\Gamma_{20C_2 r_i}(y)\Delta\Gamma_{20C_2 r_i}(y')\Big)^{\frac{1}{2}} \\
&\qquad \lesssim \delta\lambda,
\end{aligned} \tag{5.16}$$

where we used Lemma 4.1 to estimate the measure of the symmetric difference of the cones.

Now we take care of the last term in (5.15). Note that

$$|T_\mu(1_{2B_i}f)(x)| \lesssim d(x,E)^{-m-\alpha}\|1_{2B_i}f\|_{L^1(\mu)}$$

holds for every $x \in \mathbb{R}^n \setminus E$. Hence

$$\begin{aligned}\mathcal{C}^t_{\mu,20C_2r_i}(1_{2B_i}f)(y') &\lesssim \|1_{2B_i}f\|_{L^1(\mu)}\Big(\int_{\Gamma_{20C_2r_i}(y')} d(x,E)^{-2m}\,\mathrm{d}\sigma(x)\Big)^{\frac12}\\ &\lesssim \frac{\|1_{2B_i}f\|_{L^1(\mu)}}{(20C_2r_i)^m} \lesssim M_\mu f(y) \le \delta\lambda.\end{aligned}$$

Combining the above estimates with (5.14), we have shown that there exists an absolute constant C such that

$$\mathcal{C}^t_{\mu,s}(1_{(2B_i)^c}f)(y) \le (C\delta+1)\lambda.$$

If $\delta(\varepsilon)$ is small enough, then this gives (5.12).

The cases $20C_2r_i > t$ and $20C_2r_i < s$ need only parts of the above estimates. Indeed, if $20C_2r_i > t$, then as in (5.13) we have

$$\mathcal{C}^t_{\mu,s}(1_{(2B_i)^c}f)(y) \le \mathcal{C}^{20C_2r_i}_{\mu,s}(1_{(2B_i)^c}f)(y) \lesssim \delta\lambda,$$

which is clearly less than $(1+\frac{\varepsilon}{2})\lambda$ for small δ. If on the other hand $20C_2r_i < s$, then with the same estimates as above we get

$$\begin{aligned}\mathcal{C}^t_{\mu,s}(1_{(2B_i)^c}f)(y) &\le \big|\mathcal{C}^t_{\mu,s}(1_{(2B_i)^c}f)(y) - \mathcal{C}^t_{\mu,s}(1_{(2B_i)^c}f)(y')\big| + \mathcal{C}^t_{\mu,s}(1_{(2B_i)^c}f)(y')\\ &\le C\delta\lambda + \mathcal{C}^t_{\mu,s}f(y') + \mathcal{C}^t_{\mu,s}(1_{2B_i}f)(y')\\ &\le C\delta\lambda + \lambda + C\delta\lambda,\end{aligned}$$

and this again is less than $(1+\frac{\varepsilon}{2})\lambda$ for small δ. Hence (5.12) holds in any case.

Step II. Fix again some $i \in \mathcal{I}$ and consider the term

$$\mu\big(\{y \in G_i : \mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \le \delta\lambda\}\big).$$

We may assume that there exists a point y_0 in G_i such that $\mathcal{C}^t_{\mu,s}f(y_0) > (1+\varepsilon)\lambda$ and $M_\mu f(y_0) \le \delta\lambda$. Step I and the weak $(1,1)$-boundedness of $\mathcal{C}^t_{\mu,s}\colon \mathcal{M}(E) \to L^{1,\infty}(\mu\lfloor G_i)$ give

$$\begin{aligned}\mu\big(\{y \in G_i\colon \mathcal{C}^t_{\mu,s}f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \le \delta\lambda\}\big)\\ &\le \mu\big(\{y \in G_i\colon \mathcal{C}^t_{\mu,s}(1_{2B_i}f)(y) > \frac{\varepsilon}{2}\lambda\}\big)\\ &\lesssim \frac{1}{\varepsilon\lambda}\|1_{2B_i}f\|_{L^1(\mu)}\\ &\le \frac{\mu(5B_i)}{\varepsilon\lambda}M_\mu f(y_0) \le \mu(5B_i)\frac{\delta}{\varepsilon}.\end{aligned}$$

Step III. Finally we can finish the estimate (5.11). Recall that the balls $5B_i, i \in \mathcal{I}$, have bounded overlap. With Step II we have

$$\begin{aligned}\sum_{i \in \mathcal{I}} \mu\big(\{y \in G_i : \mathcal{C}^t_{\mu,s} f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \le \delta\lambda\}\big) &\lesssim \frac{\delta}{\varepsilon} \sum_{i \in \mathcal{I}} \mu(5B_i) \\ &\lesssim \frac{\delta}{\varepsilon} \mu(\Omega_\lambda).\end{aligned}$$

Combining this with Equation (5.11), we see that again if $\delta(\varepsilon)$ is small enough, then

$$\mu\big(\{y \in E \colon \mathcal{C}^t_{\mu,s} f(y) > (1+\varepsilon)\lambda; M_\mu f(y) \le \delta\lambda\}\big) \le \Big(1 - \frac{\theta}{4b}\Big)\mu(\Omega_\lambda).$$

This concludes the proof. $\square$

6. THE BIG PIECES GLOBAL Tb THEOREM

6.1. **Theorem.** *Let $B \subset E$ be a closed ball in E and assume μ is a finite Borel measure with support in B. Let $\mathcal{D}_B(\omega), \omega \in \Omega$, be a family of dyadic lattices related to the ball B as explained in Section 2. Assume $b \in L^\infty(\mu)$. Suppose $c_{acc} > 0$ and for any $\omega \in \Omega$ let T_ω be the union of the maximal cubes $R \in \mathcal{D}_B(\omega)$ that satisfy*

$$\Big| \int_R b \, \mathrm{d}\mu \Big| < c_{acc} \mu(R).$$

Furthermore, we assume that there exists a Borel set H, an exponent $s > 0$ and constants $\delta_0 \in (0,1)$ and $C_0, C_1 > 0$ so that the following conditions hold:

(1) $\mu(T_\omega \cup H) \le \delta_0 \mu(B)$ for all $\omega \in \Omega$.
(2) If B_r is a closed ball of radius r in E and $\mu(B_r) \ge C_0 r^m$, then $B_r \subset H$.
(3) $\sup_{\lambda > 0} \lambda^s \mu\big(\{y \in B \setminus H : C_\mu b(y) > \lambda\}\big) \le C_1 \mu(B)$.

Under these assumptions there exists a set $G \subset B$ with $\mu(G) \ge \frac{1-\delta_0}{3}\mu(B)$ such that

$$\|1_G \mathcal{C}_\mu f\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\mu)} \tag{6.2}$$

holds for all $f \in L^2(\mu)$.

Before the proof we recall the *b–adapted martingales*. Suppose we are in the set-up of Theorem 6.1 and let $\omega \in \Omega$. A cube $Q \in \mathcal{D}_B(\omega)$ is said to be (ω-) *transit* if $Q \not\subset T_\omega \cup H$. Denote the collection of transit cubes by $\mathcal{D}^{tr}_B(\omega)$. If g is locally μ-integrable and $Q \in \mathcal{D}_B(\omega)$ we denote the average of g over Q by

$$\langle g \rangle_Q := \frac{1}{\mu(Q)} \int_Q g \, \mathrm{d}\mu$$

with the understanding that if $\mu(Q) = 0$ then $\langle g \rangle_Q = 0$.

Let $f \in L^2(\mu)$. For the top cube $Q_B(\omega)$ define

$$E_{Q_B(\omega)} f := \frac{\langle f \rangle_{Q_B(\omega)}}{\langle b \rangle_{Q_B(\omega)}} b 1_{Q_B(\omega)}.$$

For any cube $Q \in \mathcal{D}_B^{tr}(\omega)$ define

$$\Delta_Q f := \sum_{Q' \in ch(Q)} A_{Q'} 1_{Q'},$$

where

$$A_{Q'} := \begin{cases} \Big(\frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'}} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \Big) b, & \text{if } Q' \in \mathcal{D}_B^{tr}(\omega), \\ f - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} b, & \text{if } Q' \notin \mathcal{D}_B^{tr}(\omega). \end{cases}$$

Here all the averages are defined with respect to the measure μ.

With these the $L^2(\mu)$-norm of f can be estimated as

$$\|f\|_{L^2(\mu)} \sim \Big(\|E_{Q_B(\omega)} f\|_{L^2(\mu)}^2 + \sum_{Q \in \mathcal{D}_B^{tr}(\omega)} \|\Delta_Q f\|_{L^2(\mu)}^2 \Big)^{\frac{1}{2}}, \tag{6.3}$$

and the function f can be represented as

$$f = E_{Q_B(\omega)} f + \sum_{Q \in \mathcal{D}_B^{tr}(\omega)} \Delta_Q f,$$

where convergence takes place unconditionally in $L^2(\mu)$. Also, for every $Q \in \mathcal{D}_B^{tr}(\omega)$ it holds that

$$\int_Q \Delta_Q f \, \mathrm{d}\mu = 0.$$

For a proof of these facts see Tolsa's book [20], Section 5.4.4. Notice that therein the proofs are given for the standard dyadic cubes in the plane, but the same arguments can be carried out in our setting.

Recall that the dyadic lattices $\mathcal{D}_B(\omega)$ are subcollections of dyadic lattices $\mathcal{D}(\omega)$ in E, see Section 2. Fix some $\omega_0 \in \Omega$ and write $\mathcal{D}_0 := \mathcal{D}(\omega_0)$. A cube $R \in \mathcal{D}_0$ is called ω-transit if $R \not\subset T_\omega \cup H$. Denote the collection of transit cubes in $\mathcal{D}_0$ by $\mathcal{D}_0^{tr}$. So the definition of transit cubes in $\mathcal{D}_0$ depends on ω, but later it will be clear what the ω is.

In the Tb-argument we use the following lemma, whose proof is again essentially given in [20, Lemma 5.16].

6.4. **Lemma.** *Suppose $\omega \in \Omega$ and $s > 0$. Let $\mathcal{D}_0^{tr}$ be the collection of ω-transit cubes in $\mathcal{D}_0$. For $R \in \mathcal{D}_0^{tr}$ and $Q \in \mathcal{D}_B^{tr}(\omega)$ define the numbers*

$$A_{Q,R}^s := \frac{\ell(Q)^{\frac{s}{2}} \ell(R)^{\frac{s}{2}}}{D(Q,R)^{m+s}} \mu(Q)^{\frac{1}{2}} \mu(R)^{\frac{1}{2}},$$

where $D(Q,R) := \ell(Q) + \ell(R) + d(Q,R)$. Then the matrix $\{A_{Q,R}^s\}_{Q \in \mathcal{D}_B^{tr}(\omega), R \in \mathcal{D}_0^{tr}}$ defines a bounded linear operator in ℓ^2, that is, for any two sets $\{x_Q\}_{Q \in \mathcal{D}_B^{tr}(\omega)}$, $\{y_R\}_{R \in \mathcal{D}_0^{tr}}$ of non-negative real numbers we have the estimate

$$\sum_{Q \in \mathcal{D}_B^{tr}(\omega), R \in \mathcal{D}_0^{tr}} A_{Q,R}^s x_Q y_R \lesssim \Big(\sum_{Q \in \mathcal{D}_B^{tr}(\omega)} x_Q^2 \Big)^{\frac{1}{2}} \Big(\sum_{R \in \mathcal{D}_0^{tr}} y_R^2 \Big)^{\frac{1}{2}}.$$

Now we move on to the proof of Theorem 6.1.

Proof of Theorem 6.1. Let us first give an overview of the argument. The proof is based on the idea of *suppression*. First, we suppress the square function and denote it by $\tilde{\mathcal{C}}_\mu$, so that $\tilde{\mathcal{C}}_\mu b \in L^\infty(\mu)$ and $\tilde{\mathcal{C}}_\mu f(y) = \mathcal{C}_\mu f(y)$ for every $f \in L^2(\mu)$ and y in a set G with $\mu(G) \gtrsim \mu(B)$. Moreover, this set G will be in a suitable (probabilistic) way outside of the sets $T_\omega \cup H$, in the complement of which μ is of order m and b is accretive. The nice properties of the set G combined with $\tilde{\mathcal{C}}_\mu b \in L^\infty(\mu)$ allow us to run a Tb argument and show that

$$\|1_G \tilde{\mathcal{C}}_\mu f\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\mu)}, \quad f \in L^2(\mu).$$

Since $\tilde{\mathcal{C}}_\mu f = \mathcal{C}_\mu f$ in G, we find the set we were after.

Next we present the details of the proof and divide the argument into a few steps.

Suppression. Let $\lambda_0 > 0$ be a big enough number to be specified later, and consider the set

$$S_0 := \{y \in B \colon \mathcal{C}_\mu b(y) > \lambda_0\}.$$

For every $y \in B$ define the numbers

$$t(y) := \sup\{t > 0 : \mathcal{C}_{\mu,t} b(y) > \lambda_0\}$$

and

$$r(y) := \sup\{r > 0 : \mu(B(y,r)) \ge 11^m C_0 r^m\},$$

with the convention that supremum over the empty set is zero. Since

$$\mathcal{C}_{\mu,t} b(y) \lesssim \|b\|_{L^1(\mu)} \Big(\int_{\Gamma_t(y)} d(x,E)^{-2m} \, \mathrm{d}\sigma(x) \Big)^{\frac{1}{2}} \to 0,$$

as $t \to \infty$, it is clear that $t(y)$ is finite for every $y \in B$. Notice also that $r(y)$ is finite because μ is finite.

Suppose $y \in S_0$ is such that $t(y) \ge r(y)$. By definition we have

$$\mathcal{C}_{\mu,t(y)/2} b(y) > \lambda_0.$$

We claim that

$$\mathcal{C}_{\mu,100t(y)} b(y) > \lambda_0/2$$

if λ_0 is big enough, which follows from showing that

$$\mathcal{C}^{100t(y)}_{\mu,t(y)/2} b(y) \lesssim 1.$$

If $x \in \Gamma_{t(y)/2}(y)$, then

$$\begin{aligned} |T_\mu b(x)| &\lesssim \int_E \frac{|b(z)|}{|x-z|^{m+\alpha}} \, \mathrm{d}\mu(z) \lesssim \int_E \frac{\mathrm{d}\mu(z)}{(d(x,E) + |y-z|)^{m+\alpha}} \\ &\lesssim \sum_{k=0}^{\infty} \frac{\mu\big(B(y, 2^k d(x,E))\big)}{(2^k d(x,E))^{m+\alpha}} \\ &\lesssim d(x,E)^{-\alpha}. \end{aligned} \tag{6.5}$$

In the last step it was important to observe that $d(x, E) \geq t(y)/2 \geq r(y)/2$ and

$$\mu(B(y, r)) \lesssim C_0 r^m$$

holds for all $r \geq \frac{r(y)}{2}$. Therefore Lemma 2.3 gives that

$$\mathcal{C}_{\mu,t(y)/2}^{100t(y)} b(y) \lesssim \sigma\big(\Gamma_{t(y)/2}^{100t(y)}(y)\big)^{\frac{1}{2}} \lesssim 1.$$

Consider again some $y \in S_0$ such that $t(y) \geq r(y)$, and let $y' \in B_E(y, 10t(y))$. If $x \in \Gamma_{100t(y)}(y')$, then

$$\begin{aligned} |T_\mu b(x)| \lesssim \int_E \frac{\mathrm{d}\mu(z)}{(d(x,E) + |y'-z|)^{m+\alpha}} &\sim \int_E \frac{\mathrm{d}\mu(z)}{(d(x,E) + |y-z|)^{m+\alpha}} \\ &\lesssim d(x, E)^{-\alpha} \end{aligned}$$

by Equation (6.5). Thus

$$|\mathcal{C}_{\mu,100t(y)} b(y') - \mathcal{C}_{\mu,100t(y)} b(y)|^2 \lesssim \sigma\big(\Gamma_{100t(y)}(y')\Delta\Gamma_{100t(y)}(y)\big) \lesssim 1$$

by Lemma 4.1. Hence we have shown that if λ_0 is large enough, then

$$B_E(y, 10t(y)) \subset \{y \in E \colon \mathcal{C}_\mu b(y) > \lambda_0/4\} \tag{6.6}$$

holds for all $y \in S_0$ with $t(y) \geq r(y)$.

Suppose then $y \in B$ and $r(y) > 0$, and let $y' \in B_E(y, 10r(y))$. Then it holds that

$$\frac{\mu\big(B_E(y', 11r(y))\big)}{(11r(y))^m} \geq 11^{-m} \frac{\mu\big(B_E(y, r(y))\big)}{r(y)^m} \geq C_0,$$

which shows that $B_E(y, 10r(y)) \subset H$.

Now we define the suppression. Let $y \in S_0$. If $t(y) \geq r(y)$, we define

$$A_y := \{x \in \Gamma(y) : d(x, E) < 2t(y)\},$$

and if $t(y) < r(y)$, we define

$$A_y := \{x \in \Gamma(y) : d(x, E) < r(y)\}.$$

We also set

$$A := \bigcup_{y \in S_0} A_y.$$

Since every A_y is open, we see that A is open and hence a Borel measurable subset of $\mathbb{R}^n \setminus E$. The suppressed kernel $\tilde{S}$ is defined as

$$\tilde{S}(x, y) := S(x, y) 1_{\mathbb{R}^n \setminus A}(x).$$

This is clearly a square function kernel satisfying the same size and y-Hölder conditions, and with it we define the corresponding operators $\tilde{T}_\mu$ and $\tilde{\mathcal{C}}_\mu$. From the definition it follows that

$$\tilde{\mathcal{C}}_\mu f(y) = \Big(\int_{\Gamma(y)\setminus A} |T_\mu f(x)|^2 d(x, E)^{2\alpha} \, \mathrm{d}\sigma(x)\Big)^{\frac{1}{2}} \tag{6.7}$$

for all $f \in \bigcup_{p \in [1,\infty]} L^p(\mu)$ and $y \in E$.

Next we verify the relevant properties of the suppressed operator. To this end, define the exceptional set

$$S := \bigcup_{y \in S_0} B_E\big(y, 10\max(t(y), r(y))\big).$$

Because $B_E(y, 10r(y)) \subset H$ for every y such that $r(y) > 0$, it holds that

$$S \setminus H = \bigcup_{\substack{y \in S_0 \\ t(y) \geq r(y)}} B_E(y, 10t(y)) \setminus H \subset \{y \in E \setminus H \colon \mathcal{C}_\mu b(y) > \lambda_0/4\}$$

by Equation (6.6). This implies by the weak type assumption (3) in Theorem 6.1 that

$$\mu(S \setminus H) \leq \frac{C_1 4^s}{\lambda_0^s}\mu(B) \leq \frac{1-\delta_0}{2}\mu(B)$$

if λ_0 is again big enough. We now fix a λ_0 that satisfies all the above properties, whence

$$\mu(H \cup T_\omega \cup S) \leq \mu(T_\omega \cup H) + \mu(S \setminus H) \leq \frac{1+\delta_0}{2}\mu(B). \tag{6.8}$$

We claim that

$$\tilde{\mathcal{C}}_\mu b(y) \leq \lambda_0, \quad y \in B. \tag{6.9}$$

Indeed, it is clear from (6.7) that

$$\tilde{\mathcal{C}}_\mu f(y) \leq \mathcal{C}_\mu f(y)$$

for every $f \in \bigcup_{p \in [1,\infty]} L^p(\mu)$ and $y \in B$. Thus, by the definition of S_0 we have $\tilde{\mathcal{C}}_\mu b(y) \leq \lambda_0$ for every $y \in B \setminus S_0$, while, if $y \in S_0$,

$$\tilde{\mathcal{C}}_\mu b(y) \leq \Big(\int_{\Gamma(y) \setminus A_y} |T_\mu b(x)|^2 d(x,E)^{2\alpha} \, \mathrm{d}\sigma(x) \Big)^{\frac{1}{2}} \leq \lambda_0$$

by the definitions of A_y and $t(y)$.

We shall now prove that for every f we have

$$\tilde{\mathcal{C}}_\mu f(y) = \mathcal{C}_\mu f(y), \quad y \in B \setminus S. \tag{6.10}$$

This follows from showing that $\Gamma(y) \cap A = \emptyset$ for every $y \in B \setminus S$. To get a contradiction, suppose there exist $y \in B \setminus S$ and $x \in \Gamma(y) \cap A$. Thus, there is $y' \in S_0$ so that $x \in A_{y'}$. Suppose first $t(y') \geq r(y')$. Then the definition of $A_{y'}$ implies that

$$|y - y'| \leq |y - x| + |x - y'| \leq 4d(x,E) < 8t(y'),$$

which is a contradiction because $y \notin B_E(y', 10t(y'))$. Similarly we arrive at a contradiction in the case $t(y') < r(y')$. Hence we have shown that (6.10) holds.

We remark here that (6.9) and (6.10) are still valid if we replace $\tilde{\mathcal{C}}_\mu$ by $\tilde{\mathcal{C}}^t_{\mu,s}$ and $\mathcal{C}_\mu$ by $\mathcal{C}^t_{\mu,s}$, $0 < s < t$.

The set G**.** Fix two numbers $k, l \in \mathbb{Z}, k < l$, and write $s = \delta^l$, $t = \delta^k$. Assume l is so big that $s := \delta^l < \ell(Q_B(\omega))$ for every $\omega \in \Omega$. We consider the truncated square function $\mathcal{C}^t_{\mu,s}$ and prove the required estimate (6.2) for this truncated operator with a bound that is independent on s and t. This will then finish the proof. Now that the parameter s is fixed, we also we fix the probability space $\Omega^l_{\log_\delta \ell(Q_B(\omega))} =: \widetilde{\Omega}$ (see Section 2).

For every $y \in B$ define

$$p_0(y) := \mathbb{P}\big(\{\omega \in \widetilde{\Omega} \colon y \in B \setminus (H \cup T_\omega \cup S)\}\big).$$

The set G that we are after is defined as

$$G := \{y \in B \colon p_0(y) > \tau\}$$

with some $\tau > 0$ that will be specified soon. In other words, the set G consists of those points in B that have quantitatively big probability of being outside the sets $H \cup T_\omega \cup S$.

Note that since the probability space $\widetilde{\Omega}$ consists of only finitely many points, there is no problem with measurability when defining p_0 and G.

To estimate from below the measure of G, note that by Fubini and (6.8),

$$\begin{aligned}
\int_B p_0(y)\,\mathrm{d}\mu(y) &= \int_B \int_{\widetilde{\Omega}} 1_{\{(\omega,y)\in\widetilde{\Omega}\times E\colon y\in B\setminus(H\cup T_\omega\cup S)\}}(y,\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}\mu(y) \\
&= \int_{\widetilde{\Omega}} \int_B 1_{\{(\omega,y)\in\widetilde{\Omega}\times E\colon y\in B\setminus(H\cup T_\omega\cup S)\}}(y,\omega)\,\mathrm{d}\mu(y)\,\mathrm{d}\mathbb{P}(\omega) \\
&= \int_{\widetilde{\Omega}} \mu(B \setminus (H \cup T_\omega \cup S))\, d\mathbb{P}(\omega) \\
&\geq \frac{1-\delta_0}{2}\mu(B).
\end{aligned}$$

On the other hand,

$$\int_{B\setminus G} p_0(y)\,\mathrm{d}\mu(y) \leq \tau\mu(B).$$

Therefore, since $p_0(y) \leq 1$ for all $y \in B$, we have that

$$\frac{1-\delta_0}{2}\mu(B) \leq \int_B p_0(y)\,\mathrm{d}\mu(y) \leq \tau\mu(B) + \mu(G).$$

If we set $\tau := \frac{1-\delta_0}{6}$ we infer that $\frac{1-\delta_0}{3}\mu(B) \leq \mu(G)$.

Notice also that if $h \colon E \to [0,\infty)$ is a Borel function, then

$$\begin{aligned}
\int_G h(y)\,\mathrm{d}\mu(y) &\leq \tau^{-1}\int_G h(y)p_0(y)\,\mathrm{d}\mu(y) \\
&= \tau^{-1}\int_{\widetilde{\Omega}}\int_{G\setminus(H\cup T_\omega\cup S)} h(y)\,\mathrm{d}\mu(y)\,\mathrm{d}\mathbb{P}(\omega) \qquad (6.11)\\
&= \tau^{-1}\mathbb{E}_\omega \int_{G\setminus(H\cup T_\omega\cup S)} h(y)\,\mathrm{d}\mu(y),
\end{aligned}$$

where in the last line we just denoted the integral over $\widetilde{\Omega}$ by $\mathbb{E}_\omega$.

Moreover, since $G \subset B \setminus S$, it holds that

$$\|1_G\, \mathcal{C}^t_{\mu,s} f\|^2_{L^2(\mu)} = \|1_G\, \tilde{\mathcal{C}}^t_{\mu,s} f\|^2_{L^2(\mu)}$$

for all $f \in L^2(\mu)$. For the rest of the proof we fix a function $f \in L^2(\mu)$ and show that

$$\|1_G\, \tilde{\mathcal{C}}^t_{\mu,s} f\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\mu)}.$$

Let $\omega \in \widetilde{\Omega}$ and note that $Q_B(\omega)$ must be a transit cube because $T_\omega \cup H \subsetneq B \subset Q_B(\omega)$. Hence

$$\Big|\frac{\langle f\rangle_{Q_B(\omega)}}{\langle b\rangle_{Q_B(\omega)}}\Big| \big\|1_G \tilde{\mathcal{C}}_\mu(b 1_{Q_B(\omega)})\big\|_{L^2(\mu)} \lesssim |\langle f\rangle_{Q_B(\omega)}|\mu(B)^{\frac{1}{2}} \lesssim \|f\|_{L^2(\mu)},$$

where we used the fact that $\tilde{\mathcal{C}}_\mu b \in L^\infty(\mu)$. Thus, when we represent the function f with martingale differences below, we may suppose that $\int f\, d\mu = 0$.

A probabilistic reduction. Here we make a certain probabilistic argument that corresponds to the reduction into good Whitney regions in [12]. Let $\omega \in \widetilde{\Omega}$. Using (6.11) we have

(6.12)
$$\begin{aligned}
&\|1_G \tilde{\mathcal{C}}^t_{\mu,s} f(y)\|^2_{L^2(\mu)} \\
&\quad\le \tau^{-1}\mathbb{E}_\omega \int_{G\setminus(H\cup T_\omega\cup S)} \sum_{\substack{R\in\mathcal{D}_0\\ s<\ell(R)\le t}} 1_R(y) \int_{\Gamma^{\ell(R)}_{\ell(R)\delta}(y)} |\widetilde{T}_\mu f(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y).
\end{aligned}$$

Consider some big enough goodness parameter r as in Lemma 2.11, and recall the bound $\delta^{\gamma r\eta}$ for the probability of badness. The sum over the dyadic cubes R in (6.12) may be divided into $\mathcal{D}_B(\omega)$-good and -bad cubes. The corresponding term with bad cubes only satisfies

$$\begin{aligned}
&\mathbb{E}_\omega \int_{G\setminus(H\cup T_\omega\cup S)} \sum_{\substack{R\in\mathcal{D}_0\\ s<\ell(R)\le t\\ R \text{ is } \mathcal{D}_B(\omega)\text{-bad}}} 1_R(y) \int_{\Gamma^{\ell(R)}_{\ell(R)\delta}(y)} |\widetilde{T}_\mu f(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y) \\
&\le \int_G \sum_{\substack{R\in\mathcal{D}_0\\ s<\ell(R)\le t}} \mathbb{E}_\omega 1_{\{\omega\in\widetilde{\Omega}:\ R \text{ is } \mathcal{D}_B(\omega)\text{-bad}\}}(\omega) 1_R(y) \int_{\Gamma^{\ell(R)}_{\ell(R)\delta}(y)} |\widetilde{T}_\mu f(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y) \\
&\lesssim \delta^{\gamma r\eta} \int_G \sum_{\substack{R\in\mathcal{D}_0\\ s<\ell(R)\le t}} 1_R(y) \int_{\Gamma^{\ell(R)}_{\ell(R)\delta}(y)} |\widetilde{T}_\mu f(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y) \\
&= \delta^{\gamma r\eta}\|1_G \tilde{\mathcal{C}}^t_{\mu,s} f\|^2_{L^2(\mu)}.
\end{aligned}$$

Hence, letting the goodness parameter r to be big enough, we get

(6.13)
$$\begin{aligned}
&\|1_G\tilde{\mathcal{C}}^t_{\mu,s}f\|^2_{L^2(\mu)} \\
&\quad\lesssim \mathbb{E}_\omega \int_{G\setminus(H\cup T_\omega\cup S)} \sum_{\substack{R\in\mathcal{D}_0\\ s<\ell(R)\le t\\ R\text{ is }\mathcal{D}_B(\omega)\text{-good}}} 1_R(y)\int_{\Gamma^{\ell(R)}_{\ell(R)\delta}(y)} |\widetilde{T}_\mu f(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y)\\
&\quad\le \mathbb{E}_\omega \int_B \sum_{\substack{R\in\mathcal{D}_0\\ s<\ell(R)\le t\\ R\text{ is }\mathcal{D}_B(\omega)\text{-good}\\ R\not\subset T_\omega\cup H}} 1_R(y)\int_{\Gamma^{\ell(R)}_{\ell(R)\delta}(y)} |\widetilde{T}_\mu f(x)|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y).
\end{aligned}$$

A Tb-argument. Fix some random parameter $\omega\in\widetilde{\Omega}$. Now that ω is fixed, we denote by $\mathcal{D}^{tr}_0$ the collection of ω-transit cubes in $\mathcal{D}_0$. By (6.13) it is enough prove

$$\Big(\int_B \sum_{\substack{R\in\mathcal{D}^{tr}_0\\ s<\ell(R)\le t\\ R\text{ is }\mathcal{D}_B(\omega)\text{-good}}} 1_R(y)\ell(R)^{2\alpha}\int_{\Gamma^{\ell(R)}_{\ell(R)\delta}(y)} |\widetilde{T}_\mu f(x)|^2\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y)\Big)^{\frac12}\lesssim \|f\|_{L^2(\mu)}, \tag{6.14}$$

where we noticed that $d(x,E)\sim\ell(R)$ for every $x\in\Gamma^{\ell(R)}_{\ell(R)\delta}(y)$.

Let us abbreviate

$$\mathcal{D}' := \{R\in\mathcal{D}^{tr}_0\colon\, s<\ell(R)\le t, R\cap B\ne\emptyset, R \text{ is } \mathcal{D}_B(\omega)\text{-good}\},$$

whence we may replace the sum over the cubes R in (6.14) with the sum over $R\in\mathcal{D}'$. Using martingale differences we split the function f as $f=\sum_{Q\in\mathcal{D}^{tr}_B(\omega)}\Delta_Q f$. Then the estimate (6.14) is split into four pieces according to the relative positions of the cubes Q and R. For $R\in\mathcal{D}'$ define the following collections:

(1) $\mathscr{D}_1(R) := \{Q\in\mathcal{D}^{tr}_B(\omega)\colon\, \ell(Q)<\delta\ell(R)\}$;
(2) $\mathscr{D}_2(R) := \{Q\in\mathcal{D}^{tr}_B(\omega)\colon\, \ell(Q)\ge\delta\ell(R)$ and $d(Q,R)>\ell(R)^\gamma\ell(Q)^{1-\gamma}\}$;
(3) $\mathscr{D}_3(R) := \{Q\in\mathcal{D}^{tr}_B(\omega)\colon\, \delta\ell(R)\le\ell(Q)\le\delta^{-r}\ell(R)$ and $d(Q,R)\le\ell(R)^\gamma\ell(Q)^{1-\gamma}\}$;
(4) $\mathscr{D}_4(R) := \{Q\in\mathcal{D}^{tr}_B(\omega)\colon\, \ell(Q)>\delta^{-r}\ell(R)$ and $d(Q,R)\le\ell(R)^\gamma\ell(Q)^{1-\gamma}\}$.

Next, we record with proof a few preliminary (and completely standard) estimates related to these collections. For these, recall the coefficients $A^s_{Q,R}$ from Lemma 6.4.

6.15. **Lemma.** *If $R\in\mathcal{D}'$ and $Q\in\mathscr{D}_1(R)$, then for $y\in R$ and $x\in\Gamma^{\ell(R)}_{\delta\ell(R)}(y)$ we have*

$$\ell(R)^\alpha\big|\widetilde{T}_\mu\Delta_Q f(x)\big|\lesssim A^\beta_{Q,R}\mu(R)^{-\frac12}\|\Delta_Q f\|_{L^2(\mu)}.$$

Proof. Note that in this case it holds that

$$d(x,Q)\ge d(x,E)\ge\delta\ell(R)>24\ell(Q)\ge 2\operatorname{diam}(Q).$$

Since $\Delta_Q f$ has integral zero, we can estimate

$$\begin{aligned}\ell(R)^\alpha\big|\widetilde{T}_\mu \Delta_Q f(x)\big| &= \ell(R)^\alpha\Big|\int_Q \big(\tilde{S}(x,z)-\tilde{S}(x,c_Q)\big)\Delta_Q f(z)\,\mathrm{d}\mu(z)\Big| \\ &\lesssim \ell(R)^\alpha \frac{\operatorname{diam}(Q)^\beta}{d(x,Q)^{m+\alpha+\beta}}\|\Delta_Q f\|_{L^1(\mu)} \\ &\sim \frac{\ell(R)^\alpha \ell(Q)^\beta}{\big(|x-y|+d(y,Q)\big)^{m+\alpha+\beta}}\|\Delta_Q f\|_{L^1(\mu)} \\ &\lesssim \frac{\ell(Q)^{\frac{\beta}{2}}\ell(R)^{\frac{\beta}{2}}}{\big(\ell(Q)+\ell(R)+d(Q,R)\big)^{m+\beta}}\mu(Q)^{\frac12}\|\Delta_Q f\|_{L^2(\mu)}.\end{aligned}$$

□

6.16. **Lemma.** *If $R\in\mathcal{D}'$ and $Q\in\mathscr{D}_2(R)$, then for $y\in R$ and $x\in\Gamma^{\ell(R)}_{\delta\ell(R)}(y)$ we have*

$$\ell(R)^\alpha\big|\widetilde{T}_\mu \Delta_Q f(x)\big| \lesssim A^\alpha_{Q,R}\mu(R)^{-\frac12}\|\Delta_Q f\|_{L^2(\mu)}.$$

Proof. First do the direct estimate

$$\ell(R)^\alpha\big|\widetilde{T}_\mu \Delta_Q f(x)\big| \lesssim \frac{\ell(R)^\alpha\|\Delta_Q f\|_{L^1(\mu)}}{d(x,Q)^{m+\alpha}} \lesssim \frac{\ell(R)^\alpha\|\Delta_Q f\|_{L^1(\mu)}}{\big(\ell(R)+d(Q,R)\big)^{m+\alpha}}.$$

If $\ell(Q)<d(Q,R)$, then $d(Q,R)\sim D(Q,R)$, and accordingly

$$\frac{\ell(R)^\alpha}{\big(\ell(R)+d(Q,R)\big)^{m+\alpha}} \lesssim \frac{\ell(Q)^{\frac{\alpha}{2}}\ell(R)^{\frac{\alpha}{2}}}{D(Q,R)^{m+\alpha}}.$$

On the other hand if $\ell(Q)\ge d(Q,R)$, then

$$\begin{aligned}\frac{\ell(R)^\alpha}{\big(\ell(R)+d(Q,R)\big)^{m+\alpha}} \le \frac{\ell(R)^\alpha}{\big(\ell(R)^\gamma\ell(Q)^{1-\gamma}\big)^{m+\alpha}} &= \frac{\ell(R)^{\alpha-\gamma(m+\alpha)}\ell(Q)^{\gamma(m+\alpha)}}{\ell(Q)^{m+\alpha}} \\ &\sim \frac{\ell(R)^{\frac{\alpha}{2}}\ell(Q)^{\frac{\alpha}{2}}}{D(Q,R)^{m+\alpha}},\end{aligned}$$

where we took into account that $\gamma(m+\alpha)=\frac{\alpha}{2}$. Combining these estimates proves the claim. □

The proof of the next lemma is just a direct application of the kernel size estimate.

6.17. **Lemma.** *If $R\in\mathcal{D}'$ and $Q\in\mathscr{D}_3(R)$, then for any $y\in E$ and $x\in\Gamma^{\ell(R)}_{\delta\ell(R)}(y)$ there holds that*

$$\ell(R)^\alpha\big|\widetilde{T}_\mu \Delta_Q f(x)\big| \lesssim \frac{\mu(Q)^{\frac12}\|\Delta_Q f\|_{L^2(\mu)}}{\ell(R)^m} \sim A^\alpha_{Q,R}\mu(R)^{-\frac12}\|\Delta_Q f\|_{L^2(\mu)}.$$

Using the four collections the left hand side of (6.14) satisfies $LHS(6.14) \le \sum_{i=1}^4 \Lambda_i$, where

$$\Lambda_i := \Big(\int_B \sum_{R\in\mathcal{D}'} 1_R(y)\ell(R)^{2\alpha} \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \big|\widetilde{T}_\mu \sum_{Q\in\mathscr{D}_i(R)} \Delta_Q f(x)\big|^2 \,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y)\Big)^{\frac12}.$$

Since $\sigma\big(\Gamma^{\ell(R)}_{\delta\ell(R)}(y)\big) \lesssim 1$ for every $y\in E$ and $R\in\mathcal{D}'$, from the lemmas above it is seen that

$$\begin{aligned}\Lambda_1+\Lambda_2+\Lambda_3 &\lesssim \Big(\sum_{R\in\mathcal{D}'} \Big[\sum_{Q\in\mathcal{D}^{tr}_B(\omega)} A^\beta_{Q,R}\|\Delta_Q f\|_{L^2(\mu)}\Big]^2\Big)^{\frac12} \\ &\quad + \Big(\sum_{R\in\mathcal{D}'} \Big[\sum_{Q\in\mathcal{D}^{tr}_B(\omega)} A^\alpha_{Q,R}\|\Delta_Q f\|_{L^2(\mu)}\Big]^2\Big)^{\frac12} \lesssim \|f\|_{L^2(\mu)},\end{aligned}$$

where we applied Lemma 6.4 and Equation (6.3) in the last step.

It only remains to estimate Λ_4. Suppose $R\in\mathcal{D}'$ and $k\in\mathbb{Z}$, $k\ge r$, are such that $\delta^{-k}\ell(R)\le \ell(Q_B(\omega))$. Since $R\cap Q_B(\omega)\ne\emptyset$ (as $R\cap B\ne\emptyset$ by the definition of $\mathcal{D}'$), there exists a cube $Q\in\mathcal{D}_B(\omega)$ with $\ell(Q)=\delta^{-k}\ell(R)$ so that $R\cap Q\ne\emptyset$. Then, because R is $\mathcal{D}_B(\omega)$-good, $k\ge r$ and $d(R,Q)=0\le \ell(R)^\gamma\ell(Q)^{1-\gamma}$, it must be that $d(R,E\setminus Q)>\ell(R)^\gamma\ell(Q)^{1-\gamma}$, and in particular $R\subset Q$. We denote this unique cube $Q\in\mathcal{D}_B(\omega)$ by $Q(R,k)$. Notice that $Q(R,k)$ is transit, since R is.

Note that $\mathscr{D}_4(R)$ can be non-empty only for those $R\in\mathcal{D}'$ such that $\ell(R)<\delta^r\ell(Q_B(\omega))$, and that in this case we have (using the fact that R is good)

$$\mathscr{D}_4(R) = \Big\{ Q(R,k)\colon r<k\le \log_\delta \frac{\ell(R)}{\ell(Q_B(\omega))}\Big\}. \tag{6.18}$$

If $R\in\mathcal{D}'$ and $Q\in\mathscr{D}_4(R)$, then $Q=Q(R,k)$ for some $k>r$. For notational convenience, we denote $Q_R=Q(R,k-1)$ (which exists since $k-1\ge r$). In other words, Q_R is the unique child $Q'\in ch(Q)$ that still contains R.

6.19. **Lemma.** *Let $R\in\mathcal{D}'$ and $Q\in\mathscr{D}_4(R)$. Then for $y\in R$ and $x\in\Gamma^{\ell(R)}_{\delta\ell(R)}(y)$ we have*

$$\ell(R)^\alpha\big|\widetilde{T}_\mu(1_{Q\setminus Q_R}\Delta_Q f)(x)\big| \lesssim A^\alpha_{Q,R}\mu(R)^{-\frac12}\|\Delta_Q f\|_{L^2(\mu)}.$$

Proof. Because R is good it holds that

$$d(x,Q\setminus Q_R)\gtrsim d(y,Q\setminus Q_R)\ge \ell(R)^\gamma\ell(Q_R)^{1-\gamma},$$

and so (since $\ell(Q_R)\sim\ell(Q)+\ell(R)$ and $d(Q,R)=0$) we have

$$\begin{aligned}\ell(R)^\alpha\big|\widetilde{T}_\mu(1_{Q\setminus Q_R}\Delta_Q f)(x)\big| &\lesssim \frac{\ell(R)^\alpha\|\Delta_Q f\|_{L^1(\mu)}}{d(x,Q\setminus Q_R)^{m+\alpha}} \\ &\lesssim \frac{\ell(R)^{\alpha-\gamma(m+\alpha)}\ell(Q_R)^{\gamma(m+\alpha)}}{\ell(Q_R)^{m+\alpha}}\mu(Q)^{\frac12}\|\Delta_Q f\|_{L^2(\mu)} \\ &\lesssim \frac{\ell(R)^{\frac\alpha2}\ell(Q)^{\frac\alpha2}}{D(Q,R)^{m+\alpha}}\mu(Q)^{\frac12}\|\Delta_Q f\|_{L^2(\mu)}.\end{aligned}$$

$\square$

Using Lemma 6.19 we see that

$$\Big(\int_B \sum_{R \in \mathcal{D}'} 1_R(y) \ell(R)^{2\alpha} \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \big| \widetilde{T}_\mu \sum_{Q \in \mathscr{D}_4(R)} 1_{Q \setminus Q_R} \Delta_Q f(x) \big|^2 \, \mathrm{d}\sigma(x) \, \mathrm{d}\mu(y) \Big)^{\frac{1}{2}}$$
$$\lesssim \Big(\sum_{R \in \mathcal{D}'} \Big[\sum_{Q \in \mathcal{D}^{tr}_B(\omega)} A^\alpha_{Q,R} \| \Delta_Q f \|_{L^2(\mu)} \Big]^2 \Big)^{\frac{1}{2}} \lesssim \|f\|_{L^2(\mu)},$$

where Lemma 6.4 was applied in the last step. Therefore, it remains to bound

$$\Big(\int_B \sum_{R \in \mathcal{D}'} 1_R(y) \ell(R)^{2\alpha} \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \big| \widetilde{T}_\mu \sum_{Q \in \mathscr{D}_4(R)} 1_{Q_R} \Delta_Q f(x) \big|^2 \, \mathrm{d}\sigma(x) \, \mathrm{d}\mu(y) \Big)^{\frac{1}{2}}. \tag{6.20}$$

Fix for the moment some $R \in \mathcal{D}'$ and $Q \in \mathscr{D}_4(R)$. Recall that the cube Q_R is transit. Hence we can write

$$1_{Q_R} \Delta_Q f = B_{Q,R} b - B_{Q,R} b 1_{E \setminus Q_R},$$

where

$$B_{Q,R} := \frac{\langle f \rangle_{Q_R}}{\langle b \rangle_{Q_R}} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q}.$$

Let $y \in R$ and $x \in \Gamma^{\ell(R)}_{\delta \ell(R)}(y)$, and look at the term $\big| B_{Q,R} \widetilde{T}_\mu (b 1_{E \setminus Q_R})(x) \big|$. For every $z \in E \setminus Q_R$ we have

$$|x - z| \gtrsim |y - z| \gtrsim \ell(R)^\gamma \ell(Q_R)^{1-\gamma} + |y - z|,$$

where the fact that R is good was used. Hence

$$\big| \widetilde{T}_\mu (b 1_{E \setminus Q_R})(x) \big| \lesssim \int_{E \setminus Q_R} \frac{|b(z)|}{|x - z|^{m+\alpha}} \, \mathrm{d}\mu(z) \lesssim \int_E \frac{\mathrm{d}\mu(z)}{(\ell(R)^\gamma \ell(Q_R)^{1-\gamma} + |y - z|)^{m+\alpha}}$$
$$\lesssim (\ell(R)^\gamma \ell(Q_R)^{1-\gamma})^{-\alpha},$$

where the last estimate follows from the usual calculations as in (6.5), but it is important to use the facts that R is transit, $y \in R$ and $\ell(R)^\gamma \ell(Q_R)^{1-\gamma} \geq \ell(R)$. Because Q_R is transit there holds that

$$|B_{Q,R}| \lesssim \Big| B_{Q,R} \frac{1}{\mu(Q_R)} \int_{Q_R} b \, \mathrm{d}\mu \Big| = \big| \langle \Delta_Q f \rangle_{Q_R} \big| \leq \frac{1}{\mu(Q_R)^{\frac{1}{2}}} . \| 1_{Q_R} \Delta_Q f \|_{L^2(\mu)}.$$

A combination of these estimates gives

$$\ell(R)^\alpha \big| \widetilde{T}_\mu (1_{E \setminus Q_R} B_{Q,R} b)(x) \big| \lesssim \frac{\ell(R)^{\alpha(1-\gamma)}}{\ell(Q_R)^{\alpha(1-\gamma)}} \frac{\| 1_{Q_R} \Delta_Q f \|_{L^2(\mu)}}{\mu(Q_R)^{\frac{1}{2}}}. \tag{6.21}$$

Applying (6.21) we have

$$\begin{aligned}\int_B \sum_{R\in\mathcal{D}'} 1_R(y)\ell(R)^{2\alpha} \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \Big| \sum_{Q\in\mathscr{D}_4(R)} B_{Q,R}\widetilde{T}_\mu(1_{E\setminus Q_R}b)(x)\Big|^2 \,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y) \\ \lesssim \sum_{R\in\mathcal{D}'}\mu(R)\Big[\sum_{Q\in\mathscr{D}_4(R)} \frac{\ell(R)^{\alpha(1-\gamma)}}{\ell(Q_R)^{\alpha(1-\gamma)}} \frac{\|1_{Q_R}\Delta_Q f\|_{L^2(\mu)}}{\mu(Q_R)^{\frac{1}{2}}}\Big]^2.\end{aligned} \tag{6.22}$$

Since

$$\sum_{Q\in\mathscr{D}_4(R)} \frac{\ell(R)^{\alpha(1-\gamma)}}{\ell(Q_R)^{\alpha(1-\gamma)}} \lesssim 1$$

by (6.18), we can continue with Jensen's inequality as

$$\begin{aligned} RHS\text{(6.22)} &\lesssim \sum_{R\in\mathcal{D}'}\mu(R)\sum_{Q\in\mathscr{D}_4(R)} \frac{\ell(R)^{\alpha(1-\gamma)}}{\ell(Q_R)^{\alpha(1-\gamma)}} \frac{\|1_{Q_R}\Delta_Q f\|^2_{L^2(\mu)}}{\mu(Q_R)} \\ &= \sum_{k=r}^{\infty}\delta^{k\alpha(1-\gamma)} \sum_{\substack{Q\in\mathcal{D}^{tr}_B(\omega)\\ Q\neq Q_B(\omega)}} \frac{\|1_Q\Delta_{\widehat{Q}}f\|^2_{L^2(\mu)}}{\mu(Q)} \sum_{\substack{R\in\mathcal{D}'\\ Q(R,k)=Q}}\mu(R) \\ &\le \sum_{k=r}^{\infty}\delta^{k\alpha(1-\gamma)} \sum_{\substack{Q\in\mathcal{D}^{tr}_B(\omega)\\ Q\neq Q_B(\omega)}} \|1_Q\Delta_{\widehat{Q}}f\|^2_{L^2(\mu)} \lesssim \|f\|^2_{L^2(\mu)}.\end{aligned}$$

Combining this with (6.20) we see that the last term to be estimated is

$$I := \int_B \sum_{R\in\mathcal{D}'} 1_R(y)\ell(R)^{2\alpha} \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \Big| \sum_{Q\in\mathscr{D}_4(R)} B_{Q,R}\widetilde{T}_\mu b(x)\Big|^2 \,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y). \tag{6.23}$$

So far we have used only the properties of the kernel S, but in this part we finally apply the fact that $\tilde{\mathcal{C}}_\mu b \in L^\infty(\mu)$.

Suppose $R\in\mathcal{D}'$ is such that $\ell(R) < \delta^r\ell(Q_B(\omega))$ (i.e. the case $\mathscr{D}_4(R)\neq\emptyset$). Then

$$\begin{aligned}\Big|\sum_{Q\in\mathscr{D}_4(R)} B_{Q,R}\Big| = \Big|\sum_{\substack{Q\in\mathcal{D}^{tr}_B(\omega)\\ Q\supsetneq Q(R,r)}} \Big(\frac{\langle f\rangle_{Q_R}}{\langle b\rangle_{Q_R}} - \frac{\langle f\rangle_Q}{\langle b\rangle_Q}\Big)\Big| = \Big|\frac{\langle f\rangle_{Q(R,r)}}{\langle b\rangle_{Q(R,r)}} - \frac{\langle f\rangle_{Q_B(\omega)}}{\langle b\rangle_{Q_B(\omega)}}\Big| \\ \lesssim |\langle f\rangle_{Q(R,r)}|,\end{aligned} \tag{6.24}$$

because the cubes $Q(R,k)$, $k\ge r$, are transit and $\langle f\rangle_{Q_B(\omega)} = 0$.

Using (6.24) we have

$$
\begin{aligned}
I &= \int_B \sum_{\substack{R\in\mathcal{D}' \\ \ell(R)<\delta^r\ell(Q_B(\omega))}} 1_R(y)\ell(R)^{2\alpha} \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \Big|\frac{\langle f\rangle_{Q(R,r)}}{\langle b\rangle_{Q(R,r)}}\widetilde{T}_\mu b(x)\Big|^2 \,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y) \\
&\lesssim \int_B \sum_{Q\in\mathcal{D}^{tr}_B(\omega)} |\langle f\rangle_Q|^2 \sum_{\substack{R\in\mathcal{D}' \\ Q(R,r)=Q}} 1_R(y) \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \big|\widetilde{T}_\mu b(x)\big|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y) \\
&\le \sum_{Q\in\mathcal{D}^{tr}_B(\omega)} \big|\langle f\rangle_Q\big|^2 a_Q,
\end{aligned}
$$

where

$$
a_Q := \int_B \sum_{\substack{R\in\mathcal{D}' \\ Q(R,r)=Q}} 1_R(y) \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \big|\widetilde{T}_\mu b(x)\big|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y).
$$

The last remaining thing is to verify the dyadic Carleson condition for the numbers a_Q. Luckily, this is straightforward because of our strong testing condition $\widetilde{T}_\mu b \in L^\infty(\mu)$. Indeed, fix some $Q_0 \in \mathcal{D}_B(\omega)$. Then

$$
\begin{aligned}
\sum_{\substack{Q\in\mathcal{D}^{tr}_B(\omega) \\ Q\subset Q_0}} a_Q &\le \int_B \sum_{\substack{R\in\mathcal{D}' \\ R\subset Q_0}} 1_R(y) \int_{\Gamma^{l(R)}_{l(R)\delta}(y)} \big|\widetilde{T}_\mu b(x)\big|^2 d(x,E)^{2\alpha}\,\mathrm{d}\sigma(x)\,\mathrm{d}\mu(y) \\
&\le \int_{Q_0} \tilde{\mathcal{C}}_\mu b(y)^2\,\mathrm{d}\mu(y) \lesssim \mu(Q_0).
\end{aligned}
$$

Because the Carleson condition holds, we have

$$
\sum_{Q\in\mathcal{D}^{tr}_B(\omega)} \big|\langle f\rangle_Q\big|^2 a_Q \lesssim \|f\|^2_{L^2(\mu)}.
$$

This was the last piece in the Tb argument and concludes the proof of Theorem 6.1. □

Appendix A. A sketch of the proof of Lemma 2.11

Here we sketch the proof of Lemma 2.11 following the arguments in [2], Theorem 2.11. The constants M and L in the statement of Lemma A.1 are related to properties of E as a geometrically doubling space, and are used in the construction of random dyadic cubes.

A.1. **Lemma.** *There exist two constants $C = C(M,L) > 0$ and $\eta \in (0,1]$ so that the following holds. Fix some big enough (depending on γ) goodness parameter r. Suppose $B \subset E$ is a ball in E and construct the dyadic lattices $\mathcal{D}(\omega), \omega \in \Omega$, in E using the center of B as the fixed reference dyadic point, see Section 2. For some fixed $\omega_0 \in \Omega$ write $\mathcal{D}_0 = \mathcal{D}(\omega_0)$ and recall the lattices $\mathcal{D}_B(\omega) \subset \mathcal{D}(\omega)$.*

Assume $k_0 \in \mathbb{Z}$ is such that $\ell(Q_B(\omega)) = \delta^{k_0}$ for some, and hence for every, $\omega \in \Omega$. Let $k_1 \in \mathbb{Z}$ be any number such that $k_1 \geq k_0 + r$. With the fixed ω_0 define the probability space

$$\Omega_{k_0}^{k_1} := \{\omega \in \Omega \colon \omega(m) = \omega_0(m) \text{ if } m < k_0 \text{ or } m > k_1\}$$

equipped with the natural probability measure such that the coordinates $m \mapsto \omega(m)$, $k_0 \leq m \leq k_1$, are independent and uniformly distributed over $\{0, \dots, L\} \times \{1, \dots, M\}$.

Then, for every cube $R \in \mathcal{D}_0$ with $\ell(R) \geq \delta^{k_1}$ it holds that

$$\mathbb{P}\big(\{\omega \in \Omega_{k_0}^{k_1} : R \text{ is } \mathcal{D}_B(\omega)\text{-bad}\}\big) \leq C\delta^{\gamma r \eta}. \tag{A.2}$$

Before the proof define for $\varepsilon > 0$ the *ε-boundary* $\partial_\varepsilon Q$ of a cube $Q \in \mathcal{D}(\omega)$ by

$$\partial_\varepsilon Q := \{y \in Q \colon d(y, E \setminus Q) < \varepsilon \ell(Q)\}.$$

Proof of Lemma A.1. Let $R \in \mathcal{D}_0$ be such that $\ell(R) = \delta^m$, where $k_0 + r \leq m \leq k_1$ (if $m < k_0 + r$, then R is automatically good by definition). Fix some $\omega \in \Omega_{k_0}^{k_1}$ for the moment. First we show that if R is $\mathcal{D}_B(\omega)$-bad, then there exists $l \in \mathbb{Z}$, $k_0 \leq l \leq m - r$, and $Q \in \mathcal{D}_l(\omega)$ so that $c_R \in \partial_{7\ell(R)^\gamma/\ell(Q)^\gamma} Q$. Indeed, let $k_0 \leq l \leq m - r$ and suppose $Q \in \mathcal{D}_l(\omega)$ is such that $c_R \in Q$. If $c_R \notin \partial_{7\ell(R)^\gamma/\ell(Q)^\gamma} Q$, then because $R \subset B(c_R, 6\ell(R))$ by (2.7), it is seen that

$$d(R, E \setminus Q) \geq \ell(R)^\gamma \ell(Q)^{1-\gamma}.$$

In particular, we have

$$\max\big(d(R, Q'), d(R, E \setminus Q')\big) \geq \ell(R)^\gamma \ell(Q')^{1-\gamma}$$

for all $Q' \in \mathcal{D}_B(\omega)$ with $\ell(Q') = \delta^l$. If this happens for all l such that $k_0 \leq l \leq m - r$, then the cube R is $\mathcal{D}_B(\omega)$-good. We have shown that

$$\mathbb{P}\big(\{\omega \in \Omega_{k_0}^{k_1} : R \text{ is } \mathcal{D}_B(\omega)\text{-bad}\}\big) \leq \sum_{l=k_0}^{m-r} \mathbb{P}\Big(\Big\{\omega \in \Omega_{k_0}^{k_1} : c_R \in \bigcup_{Q \in \mathcal{D}_l(\omega)} \partial_{7\ell(R)^\gamma/\ell(Q)^\gamma} Q\Big\}\Big). \tag{A.3}$$

Next we fix some $l \in \{k_0, \dots, m - r\}$ and estimate the corresponding term in the right hand side of (A.3). Following the argument in [2], if $\omega(p)$ has a certain value for some $p \geq l$ such that $\delta^p \geq 49\delta^{m\gamma}\delta^{l(1-\gamma)}$, then

$$c_R \notin \bigcup_{Q \in \mathcal{D}_l(\omega)} \partial_{7\ell(R)^\gamma/\ell(Q)^\gamma} Q. \tag{A.4}$$

This part of the argument is short, but to state it we would need to introduce more of the construction of the random dyadic cubes. We refer the reader to [2].

The requirement $\delta^l \geq \delta^p \geq 49\delta^{m\gamma}\delta^{l(1-\gamma)}$ amounts to

$$l \leq p \leq \log_\delta 49 + m\gamma + l(1 - \gamma). \tag{A.5}$$

Note that in particular every such p satisfies $k_0 \leq p \leq k_1$. For (A.5) to make sense we demand r to be so big that $r\gamma > 2$, say, whence

$$\log_\delta 49 + m\gamma + l(1 - \gamma) - l \geq (m - l)\gamma - 1 > 1. \tag{A.6}$$

Denote by $\lfloor \log_\delta 49 + (m-l)\gamma \rfloor$ the smallest integer less than or equal to $\log_\delta 49 + (m-l)\gamma$.

For every $p \in \mathbb{Z}$ the variable $\omega(p)$ has the probability $\tau := \frac{1}{M(L+1)}$ of getting a given value. Hence, by (A.4), we have

$$\mathbb{P}\big(\{\omega \in \Omega_{k_0}^{k_1} : c_R \in \bigcup_{Q \in \mathcal{D}_l(\omega)} \partial_{7\delta^{(m-l)\gamma}} Q\}\big) \le (1-\tau)^{\lfloor \log_\delta 49 + (m-l)\gamma \rfloor} \le C(1-\tau)^{(m-l)\gamma}.$$

Combining this with (A.3) we get

$$\mathbb{P}\big(\{\omega \in \Omega_{k_0}^{k_1} : R \text{ is } \mathcal{D}_B(\omega)\text{-bad}\}\big) \le C \sum_{l=k_0}^{m-r} (1-\tau)^{(m-l)\gamma} \sim (1-\tau)^{r\gamma},$$

and we can rewrite the bound as

$$(1-\tau)^{r\gamma} = \delta^{r\gamma \log_\delta (1-\tau)}.$$

This gives the required conclusion because $\log_\delta(1-\tau) \in (0,1)$. □

Appendix B. L^2 boundedness implies weak $(1,1)$ boundedness

We verify here that if μ is a measure of order m in E and $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$, then

$$\mathcal{C}\colon \mathcal{M}(E) \to L^{1,\infty}(\mu)$$

boundedly. The proof of this follows the standard steps using the Calderón-Zygmund decomposition, but we check the details because of our unusual set-up.

First we record a few lemmas, and begin with the non-homogeneous Calderón-Zygmund decomposition whose proof can be found for example in [20]. We say that a collection $\{B_i\}_i$ of balls in $\mathbb{R}^n$ has *bounded overlap* if there exists a constant C such that

$$\sum_i 1_{B_i}(x) \le C$$

for every $x \in \mathbb{R}^n$.

B.1. **Lemma.** *Let μ be a Radon measure in $\mathbb{R}^n$ and suppose ν is a complex measure in $\mathbb{R}^n$ with compact support. Let $\lambda > 2^{n+1}\frac{|\nu|(\mathbb{R}^n)}{\mu(\mathbb{R}^n)}$.*

There exists a countable family $\{B_i\}_{i \in \mathcal{I}}$ of closed balls with bounded overlap and with centers in $\operatorname{spt}\nu$, and a function $f \in L^1(\mu)$ with $\|f\|_{L^\infty(\mu)} \le \lambda$ so that

$$|\nu|(B_i) > 2^{-n-1}\lambda\mu(2B_i) \ \text{ for all } i, \tag{B.2}$$

$$|\nu|(\eta B_i) \le 2^{-n-1}\lambda\mu(2\eta B_i) \ \text{ for all } i \text{ and } \eta > 2, \tag{B.3}$$

$$1_{\mathbb{R}^n \setminus \bigcup_i B_i}\nu = f\mu. \tag{B.4}$$

For every $i \in \mathcal{I}$ suppose R_i is $(6, 6^{m+1})$-doubling ball (with respect to μ) concentric with B_i and $r(R_i) > 4r(B_i)$. Define the functions $w_i := \frac{1_{B_i}}{\sum_j 1_{B_j}}$. Then there exists *a family* $\{\varphi_i\}_{i\in\mathcal{I}}$ *of functions such that each function is of the form* $\varphi_i = \alpha_i h_i$*, where* $\alpha_i \in \mathbb{C}$ *and* h_i *is a non-negative function, and this family satisfies the properties*

$$\operatorname{spt}\varphi_i \subset R_i, \tag{B.5}$$

$$\int \varphi_i \,\mathrm{d}\mu = \int w_i \,\mathrm{d}\nu, \tag{B.6}$$

$$\sum_{i\in\mathcal{I}} |\varphi_i| \le C_1\lambda, \tag{B.7}$$

$$\|\varphi_i\|_{L^\infty(\mu)}\mu(R_i) \le c|\nu|(B_i). \tag{B.8}$$

Here C_1 *is a constant depending on* m *and* n*, and* c *is an absolute constant.*

The next two simple lemmas can also be found for example in [20].

B.9. **Lemma.** *Suppose* μ *is a measure of order* m *in* $\mathbb{R}^n$*. Let* $b > a^m$*. If* B *is a ball in* $\mathbb{R}^n$*, then there exists* $s > 1$ *such that the ball* sB *is* (a,b)*-doubling.*

B.10. **Lemma.** *Suppose* μ *is a Radon measure in* $\mathbb{R}^n$*. Let* $b > a^m, a > 1$*. Suppose* B_1 *and* B_2 *are two balls in* $\mathbb{R}^n$ *with center* x *and* $B_1 \subset B_2$*. Assume none of the balls* $a^k B_1$ *is* (a,b)*-doubling for those* $k \in \mathbb{Z}$ *such that* $B_1 \subsetneq a^k B_1 \subset B_2$*. Then*

$$\int_{B_2\setminus B_1} \frac{1}{|y-x|^m}\,\mathrm{d}\mu(y) \lesssim_{a,b,m} \frac{\mu(B_2)}{r(B_2)^m}.$$

B.11. **Theorem.** *Let* μ *be a measure of order* m *in* E*. Suppose that* $\mathcal{C}_\mu$ *is bounded in* $L^2(\mu)$*. Then*

$$\mathcal{C}\colon \mathcal{M}(E) \to L^{1,\infty}(\mu)$$

is bounded with a constant depending on the kernel parameters, the dimension n *and the* $L^2(\mu)$ *norm of* $\mathcal{C}_\mu$*.*

Proof. Let $\nu \in \mathcal{M}(E)$ and $\lambda > 0$. We want to show that

$$\mu\big(\{y\in E\colon \mathcal{C}\nu(y) > \lambda\}\big) \lesssim \frac{1}{\lambda}|\nu|(E).$$

We may assume that $\lambda > 2^{n+1}\frac{|\nu|(E)}{\mu(E)}$, since otherwise we have nothing to prove.

Suppose first that ν is compactly supported. We apply Lemma B.1 with λ to the measures ν and μ to get a function f and an almost disjoint collection $\{B_i\}$ of closed balls with centers in E such that (B.2), (B.3) and (B.4) hold. For each i, let $R_i \supsetneq 4B_i$ be the smallest $(8, 8^{m+1})$-doubling closed ball concentric with B_i, which exists by Lemma B.9. We apply Lemma B.1 with the balls R_i to get a collection $\{\varphi_i\}$ of functions such that (B.5), (B.6), (B.7) and (B.8) hold.

Using the balls B_i and the functions φ_i we can write the measure ν as

$$\begin{aligned}\nu = 1_{\mathbb{R}^n\setminus\bigcup_i B_i}\nu + 1_{\bigcup_i B_i}\nu &= f\mu + \sum_i w_i\nu \\ &= \big(f + \sum_i \varphi_i\big)\mu + \sum_i (w_i\nu - \varphi_i\mu).\end{aligned}$$

Write $g = f + \sum_i \varphi_i$ and $b = \sum_i b_i = \sum_i (w_i \nu - \varphi_i \mu)$. Then we have

$$\mu\big(\{y \in E \colon \mathcal{C}\nu(y) > \lambda\}\big) \le \mu\big(\{y \in E \colon \mathcal{C}_\mu g > \frac{\lambda}{2}\}\big) + \mu\big(\{y \in E \colon \mathcal{C}b > \frac{\lambda}{2}\}\big). \tag{B.12}$$

The $L^2(\mu)$-boundedness of $\mathcal{C}_\mu$ and the fact that $\|f + \sum_i \varphi_i\|_{L^\infty(\mu)} \lesssim \lambda$ by (B.7) give

$$\mu\big(\{y \in E \colon \mathcal{C}_\mu g > \frac{\lambda}{2}\}\big) \lesssim \frac{1}{\lambda^2}\|f + \sum_i \varphi_i\|_{L^2(\mu)}^2 \lesssim \frac{1}{\lambda}\|f + \sum_i \varphi_i\|_{L^1(\mu)}.$$

Using Equations (B.4), (B.5), (B.8) and the bounded overlapping property of $\{B_i\}$ we get

$$\begin{aligned}\|f + \sum_i \varphi_i\|_{L^1(\mu)} \le \int |f| \, \mathrm{d}\mu + \sum_i \int |\varphi_i| \, \mathrm{d}\mu &\le |\nu|(\mathbb{R}^n) + \sum_i \|\varphi_i\|_{L^\infty(\mu)} \mu(R_i) \\ &\lesssim |\nu|(\mathbb{R}^n) + \sum_i |\nu|(B_i) \\ &\lesssim |\nu|(\mathbb{R}^n).\end{aligned}$$

Next, we consider the second term on the right hand side of (B.12). Note that

$$\mu\big(\bigcup_i 2B_i\big) \le \sum_i \mu(2B_i) \le \frac{2^{n+1}}{\lambda} \sum_i |\nu|(B_i) \lesssim \frac{1}{\lambda}|\nu|(\mathbb{R}^n).$$

Hence we need to show that

$$\mu\big(\{y \in E \setminus \bigcup_i 2B_i \colon \mathcal{C}b > \frac{\lambda}{2}\}\big) \lesssim \frac{1}{\lambda}|\nu|(\mathbb{R}^n).$$

First estimate as

$$\begin{aligned}\mu\big(\{y \in E \setminus \bigcup_i 2B_i \colon \mathcal{C}b > \frac{\lambda}{2}\}\big) &\le \frac{2}{\lambda} \int_{E \setminus \bigcup_i 2B_i} \mathcal{C}b \, \mathrm{d}\mu \\ &\le \frac{2}{\lambda} \sum_i \int_{E \setminus 2B_i} \mathcal{C}b_i \, \mathrm{d}\mu.\end{aligned}$$

We will prove that

$$\int_{E \setminus 2B_i} \mathcal{C}b_i \, \mathrm{d}\mu \lesssim |\nu|(B_i) \tag{B.13}$$

holds for every i, which then concludes the proof because $\sum_i |\nu|(B_i) \lesssim |\nu|(\mathbb{R}^n)$.

Fix some i, and recall the ball R_i related to the ball B_i. We begin the proof of (B.13) by writing

$$\int_{E \setminus 2B_i} \mathcal{C}b_i \, \mathrm{d}\mu = \int_{E \setminus 8R_i} \mathcal{C}b_i \, \mathrm{d}\mu + \int_{8R_i \setminus 2B_i} \mathcal{C}b_i \, \mathrm{d}\mu =: I + II.$$

We consider the term I first. Let $y \in E \setminus 8R_i$ and $x \in \Gamma(y)$. Let c_{R_i} be the center of R_i, whence it follows that $|x - c_{R_i}| \geq 2r(R_i)$. Since $\operatorname{spt} b_i \subset R_i$ and $b_i(R_i) = 0$, we may apply the y-continuity (1.2) of the square function kernel to get

$$|Tb_i(x)| \lesssim \frac{r(R_i)^\beta}{|x - c_{R_i}|^{m+\alpha+\beta}} |b_i|(R_i) \sim \frac{r(R_i)^\beta}{(|x-y| + |y - c_{R_i}|)^{m+\alpha+\beta}} |b_i|(R_i).$$

Also

$$\begin{aligned} |b_i|(R_i) = |w_i\nu - \varphi_i\mu|(R_i) &\leq \int_{R_i} w_i \,\mathrm{d}|\nu| + \int_{R_i} |\varphi_i| \,\mathrm{d}\mu \\ &\leq |\nu|(B_i) + \|\varphi_i\|_{L^\infty(\mu)} \mu(R_i) \lesssim |\nu|(B_i). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{C}b_i(y)^2 &\lesssim \int_{\Gamma(y)} \frac{r(R_i)^{2\beta} d(x,E)^{2\alpha}}{(|x-y| + |y - c_{R_i}|)^{2(m+\alpha+\beta)}} \,\mathrm{d}\sigma(x) |b_i|(R_i)^2 \\ &\lesssim \frac{r(R_i)^{2\beta}}{|y - c_{R_i}|^{2(m+\beta)}} |\nu|(B_i)^2, \end{aligned}$$

where we applied Lemma 2.4 in the second step. Because μ is of order m we get

$$\begin{aligned} I = \int_{E \setminus 8R_i} \mathcal{C}b_i \,\mathrm{d}\mu &\lesssim \int_{E \setminus 8R_i} \frac{r(B_i)^\beta}{|y - c_{R_i}|^{m+\beta}} \,\mathrm{d}\mu(y) |\nu|(B_i) \\ &\lesssim |\nu|(B_i)|. \end{aligned}$$

It remains to consider the term II. Since $b_i = w_i\nu - \varphi_i\mu$ we have

$$II = \int_{8R_i \setminus 2B_i} \mathcal{C}b_i \,\mathrm{d}\mu \leq \int_{8R_i \setminus 2B_i} \mathcal{C}(w_i\nu) \,\mathrm{d}\mu + \int_{8R_i \setminus 2B_i} \mathcal{C}_\mu \varphi_i \,\mathrm{d}\mu =: II_1 + II_2.$$

The $L^2(\mu)$-boundedness of $\mathcal{C}_\mu$ gives

$$II_2 \leq \mu(8R_i)^{\frac{1}{2}} \|\mathcal{C}_\mu \varphi_i\|_{L^2(\mu)} \lesssim \mu(R_i)^{\frac{1}{2}} \|\varphi_i\|_{L^2(\mu)} \leq \|\varphi_i\|_{L^\infty(\mu)} \mu(R_i) \lesssim |\nu|(B_i),$$

where we used the fact that R_i is $(8, 8^{m+1})$-doubling and the properties (B.5) and (B.8) of the function φ_i.

Finally, we consider the term II_1. Suppose $y \in 8R_i \setminus 2B_i$ and $x \in \Gamma(y)$. Then, by Lemma 2.2,

$$|T(w_i\nu)(x)| \lesssim \int_{B_i} \frac{\mathrm{d}|\nu|(z)}{|x-z|^{m+\alpha}} \sim \frac{|\nu|(B_i)}{(|x-y| + |y - c_{B_i}|)^{m+\alpha}},$$

and hence

$$\begin{aligned} \mathcal{C}(w_i\nu)(y) &\lesssim \Big(\int_{\Gamma(y)} \frac{d(x,E)^{2\alpha}}{(|x-y| + |y - c_{B_i}|)^{2(m+\alpha)}} \,\mathrm{d}\sigma(x) \Big)^{\frac{1}{2}} |\nu|(B_i) \\ &\lesssim \frac{1}{|y - c_{B_i}|^m} |\nu|(B_i). \end{aligned}$$

Integrating this over $8R_i \setminus 2B_i$ gives

$$\begin{aligned} II_1 = \int_{8R_i \setminus 2B_i} \mathcal{C}(w_i \nu)\,\mathrm{d}\mu &\lesssim \int_{8R_i \setminus R_i} \frac{1}{|y - c_{B_i}|^m}\,\mathrm{d}\mu(y)|\nu|(B_i) \\ &\quad + \int_{R_i \setminus 2B_i} \frac{1}{|y - c_{B_i}|^m}\,\mathrm{d}\mu(y)|\nu|(B_i) \\ &\lesssim |\nu|(B_i), \end{aligned}$$

where we used Lemma B.10 to estimate the integral over $R_i \setminus 2B_i$. This finishes the proof (B.13), and hence also the proof Theorem B.11 in the case when ν is compactly supported.

Suppose then ν is not compactly supported but μ is compactly supported. Suppose $M > 0$ is such that $\operatorname{spt}\mu \subset B(0, M/2)$. Write $\tilde{\nu} := \nu\lfloor(E \setminus B(0, M))$. Then for any $y \in \operatorname{spt}\mu$ and $x \in \Gamma(y)$ we have

$$\begin{aligned} \big|T\tilde{\nu}(x)\big| &\lesssim \int_{E \setminus B(0,M)} \frac{\mathrm{d}|\nu|(z)}{(|x - y| + |y - z|)^{m+\alpha}} \\ &\leq \frac{|\nu|(\mathbb{R}^n)}{\big(|x - y| + d(y, E \setminus B(0, M))\big)^{m+\alpha}}, \end{aligned}$$

and this gives by Lemma 2.4 that

$$\begin{aligned} \mathcal{C}\tilde{\nu}(y) &\lesssim \Big(\int_{\Gamma(y)} \frac{d(x, E)^{2\alpha}}{\big(|x - y| + d(y, E \setminus B(0, M))\big)^{2(m+\alpha)}}\,\mathrm{d}\sigma(x) \Big)^{\frac{1}{2}} |\nu|(\mathbb{R}^n) \\ &\lesssim \frac{1}{d(y, E \setminus B(0, M))^m} |\nu|(\mathbb{R}^n). \end{aligned}$$

Hence, if M is big enough, we get

$$\begin{aligned} \mu\big(\{y \in E \colon \mathcal{C}\nu(y) > \lambda\}\big) &\leq \mu\big(\{y \in E \colon \mathcal{C}\big(\nu\lfloor B(0, M)\big)(y) > \frac{\lambda}{2}\}\big) \\ &\lesssim \frac{1}{\lambda}|\nu|\big(B(0, M)\big) \leq \frac{1}{\lambda}|\nu|(\mathbb{R}^n), \end{aligned}$$

where the second inequality holds because $\nu\lfloor B(0, M)$ is compactly supported.

Suppose finally that neither ν nor μ is compactly supported. Then for every $M > 0$ it holds that

$$\mu\lfloor B(0, M)\big(\{y \in E \colon \mathcal{C}\nu(y) > \lambda\}\big) \lesssim \frac{1}{\lambda}|\nu|(\mathbb{R}^n),$$

because $\mu\lfloor B(0, M)$ is compactly supported. Letting M tend to infinity concludes the proof.

□

References

[1] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, C. Thiele, Carleson measures, trees, extrapolation, and T(b) theorems. Publ. Mat. 46 (2002), no. 2, 257–325.

[2] P. Auscher, T. Hytönen, Orthonormal bases of regular wavelets in spaces of homogeneous type. Appl. Comput. Harmon. Anal. 34 (2013), no. 2, 266–296.

[3] P. Auscher, E. Routin, Local Tb theorems and Hardy inequalities. J. Geom. Anal. 23 (2013), no. 1, 303–374.

[4] P. Auscher, Q. Yang, BCR algorithm and the T(b) theorem. Publ. Mat. 53 (2009), no. 1, 179–196.

[5] J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, A. Volberg, Rectifiability of harmonic measure. Geom. Funct. Anal., to appear, arXiv:1509.06294, 2015.

[6] G. David, S. Semmes, Analysis of and on uniformly rectifiable sets. Mathematical Surveys and Monographs, 38. American Mathematical Society, Providence, RI, 1993.

[7] A. Grau de la Herran, M. Mourgoglou, A local Tb theorem for square functions in domains with Ahlfors-David regular boundaries. J. Geom. Anal. 24 (2014), no. 3, 1619–1640.

[8] S. Hofmann, A proof of the local Tb theorem for standard Calderón–Zygmund operators. Unpublished manuscript, arXiv:0705.0840, 2007.

[9] S. Hofmann, D. Mitrea, M. Mitrea, A. Morris, L^p-square function estimates on spaces of homogeneous type and on uniformly rectifiable sets. Mem. Amer. Math. Soc., to appear, arXiv:1301.4943, 2013.

[10] T. Hytönen, H. Martikainen, Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces. J. Geom. Anal. 22 (2012), no. 4, 1071–1107.

[11] T. Hytönen, F. Nazarov, The local Tb theorem with rough test functions. Preprint, arXiv:1206.0907, 2012.

[12] H. Martikainen, M. Mourgoglou, Square functions with general measures. Proc. Amer. Math. Soc. 142 (2014), no. 11, 3923–3931.

[13] H. Martikainen, M. Mourgoglou, T. Orponen, Square functions with general measures II. Indiana Univ. Math. J. 63 (2014), no. 5, 1249–1279.

[14] H. Martikainen, M. Mourgoglou, X. Tolsa, Improved Cotlar's inequality in the context of local Tb theorems. Preprint, arXiv:1512.02950, 2015.

[15] H. Martikainen, M. Mourgoglou, E. Vuorinen, A new approach to non-homogeneous local Tb theorems: Square functions and weak (1,1) testing with measures. Preprint, arXiv:1511.00528, 2015.

[16] P. Mattila, Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

[17] F. Nazarov, S. Treil, A. Volberg, The Tb-theorem on non-homogeneous spaces that proves a conjecture of Vitushkin. CRM preprint (2002), 519:1–84.

[18] F. Nazarov, S. Treil, A. Volberg, The Tb-theorem on non-homogeneous spaces. Acta Math. 190 (2003), no. 2, 151–239.

[19] E. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[20] X. Tolsa, Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory. Progress in Mathematics, 307. Birkhäuser/Springer, Cham, 2014.

[21] A. Volberg, Calderón-Zygmund capacities and operators on nonhomogeneous spaces. CBMS Regional Conference Series in Mathematics, 100. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2003.

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